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Some Common Fixed Point Theorems of Contractive Mappings in Cone b-metric Spaces

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Abstract: In this paper, we prove some common fixed point theorems of contraction mappings in cone b-metric spaces.

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1. Introduction

Fixed point theory plays a basic role in application of many branches of mathematics. Finding a fixed point of contractive mapping becomes the center of strong research activity. There are many works about the fixed point of contractive maps (see, for example, [1, 2]). In [2] Polish Mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922. In [3], Bakhtin introduced b-metric spaces as a generalized of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. A lucid survey shows that there are many generalizations of metric spaces. One of them is b-metric space. The concept of b-metric space was introduced by Czerwik [11, 12]. Using this idea, he proved Banach’s fixed point theorem in b-metric spaces. Later on, many researchers including Aydi [8], Bota [9], Chug [10], Shi [20], Du [13], Kir [19], Huang and Zhang [4] introduced cone metric spaces as generalized of metric spaces, replacing the real number by an ordered Banach spaces and define cone metric spaces. Moreover, they proved some fixed point theorems for contraction mapping that expanded certain results of fixed point in metric spaces. In [5], Hussain and Shah introduced cone b-metric spaces as a generalized of b-metric spaces and cone metric spaces. We prove some fixed point theorems in cone b-metric spaces on contraction mapping.

Before going to the main results, we define some definition, example and lemma required in sequel.

2. Preliminaries

Let E be real Banach space and P be a subset of E. Then P is called a cone if

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We write \( \| \) to indicate that \( \| \) is the norm on \( E \). The cone \( P \) is called normal if there is a number \( k > 1 \) such that \( \| x \| \leq k \| y \| \) for all \( x, y \in \) int \( P \). The least positive number \( k \) satisfying the above is called the normal constant of \( P \). It is well known that \( k \geq 1 \).

In the following, we always suppose that \( E \) is a Banach space, \( P \) is cone in \( E \) with respect to \( P \).

**Definition 2.1** ([4]). Let \( X \) be a nonempty set. Suppose that the mapping \( d : X \times X \to E \) Satisfies:

(CM1) \( \theta \leq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = \theta \) if and only if \( x = y \),

(CM2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \),

(CM3) \( d(x, y) = d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a cone metric on \( X \) and \( (X, d) \) is called a cone metric space.

**Definition 2.2** ([5]). Let \( X \) be a nonempty set and \( s \geq 1 \) be a real number. Suppose that the mapping \( d : X \times X \to E \) Satisfies:

(CBM1) \( \theta \leq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = \theta \) if and only if \( x = y \),

(CBM2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \),

(CBM3) \( d(x, y) = s \| d(x, z) + d(z, y) \| \) for all \( x, y, z \in X \).

Then \( d \) is called a cone b-metric on \( X \) and \( (X, d) \) is called a cone b-metric space.

**Remark 2.3.** The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric space generalized b-metric spaces and cone metric spaces.

Following is an example which shows that a cone b-metric spaces which are not cone metric spaces:

**Example 2.4** ([7]). \( E = \mathbb{R}^2 \), \( P = \{(x, y) \in E : x, y \geq 0\} \subset E \), \( X = \mathbb{R} \) and \( d : X \times X \to E \) such that \( d(x, y) = (|x - y|^p, \alpha |x - y|^p) \) where \( \alpha \geq 0 \) and \( p > 1 \) are two constant. Then \( (X, d) \) is a cone b-metric space but not a cone metric space. In fact, we only need to prove (iii) in Definition 2.2 as follows: Let \( x, y, z \in X \). Set \( u = x - z \), \( v = z - y \), so \( x - y = u + v \) from the inequality \( (a + b)^p = (2 \max \{a, b\})^p \leq 2^p (a^p + b^p) \) for all \( a, b \geq 0 \), we have

\[ |x - y|^p = |u + v|^p \leq (|u| + |v|)^p = 2^p (|u|^p + |v|^p) = 2^p (|x - z|^p + |z - y|^p), \]

\[ \Rightarrow d(x, y) = s \|[d(x, z) + d(z, y)]\| \]

with \( s = 2^p > 1 \). But \( |x - y|^p = |x - z|^p + |z - y|^p \) is impossible for all \( x > z > y \), indeed, taking account of the inequality.

\[ (a + b)^p > a^p + b^p \text{ for all } a, b > 0, \]
We arrive at
\[ |x - y|^p = |u + v|^p = |u + v| > u^p + v^p = (x - z)^p + (z - y)^p = |x - z|^p + |z - y|^p, \]
for all \( x > z > y \). Thus, (CM3) in Definition 2.1 is not satisfied, i.e., \((X, d)\) is not a cone metric space.

**Example 2.5** ([7]). Let \( X = l^p \) with \( 0 < p < 1 \), where \( l^p = \{ \{x_n\} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty \} \). Let \( d : X \times X \rightarrow R^+ \),
\[ d(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}, \text{ where } x = \{x_n\}, y = \{y_n\} \in l^p. \]

Then \((x, d)\) is a b-metric space (see [5]). Put \( E = l^1, P = \{ \{x_n\} \in E : x_n \geq 0, \text{ for all } n \geq 1 \} \). Letting the mapping \( d : X \times X \rightarrow E \) be defined by \( d(x, y) = \left( \frac{2|x|}{2|x|^p} \right)_{n \geq 1} \), we conclude that \((X, d)\) is a cone b-metric with coefficient \( s = 2^{\frac{1}{p}} > 1 \), but it is not a cone metric space.

**Example 2.6** ([7]). Let \( X = \{1, 2, 3, 4\}, E = R^2, P = \{(x, y) \in E : x \geq 0, y \geq 0\} \). Define \( d : X \times X \rightarrow E \)
\[ d(x, y) = \begin{cases} (|x - y|^{-1}, |x - y|^{-1}) , & \text{if } x \neq y; \\ \theta , & \text{if } x = y. \end{cases} \]

Then \((X, d)\) is a cone b-metric space with the coefficient \( s = \frac{3}{2} \). But it is not a cone metric space since the triangle inequality is not satisfied. Indeed, \( d(1, 2) > d(1, 4) + d(4, 2) \) and \( d(3, 4) > d(3, 1) + d(1, 4) \).

**Definition 2.7** ([5]). Let \((X, d)\) be a cone b-metric space, \( x \in X \) and \( \{x_n\} \) be a sequence in \( X \). Then

1. \( \{x_n\} \) converges to \( x \) whenever, for every \( c \in E \) with \( \theta \ll c \), there is a natural number \( N \) such that \( d(x_n, x) \ll c \) for all \( n \geq N \). We denote this by \( \lim_{n \rightarrow \infty} x_n = x \) or \( x_n \rightarrow x \) \((n \rightarrow \infty)\).

2. \( \{x_n\} \) is a Cauchy sequence whenever, for every \( c \in E \) with \( \theta \ll c \), there is a natural number \( N \) such that \( d(x_n, x_m) \ll c \) for all \( n, m \geq N \).

3. \((X, d)\) is a complete cone b-metric space if every Cauchy sequence is convergent.

**Lemma 2.8** ([6]). Let \( P \) be a cone and \( \{x_n\} \) be a sequence in \( E \). If \( c \in \text{int} \ P \) and \( \theta \ll x_n \rightarrow \theta \) \((n \rightarrow \infty)\), then there exist \( N \) such that for all \( n > N \), we have \( x_n \ll c \).

**Lemma 2.9** ([6]). Let \( x, y, z \in E \), if \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

**Lemma 2.10** ([5]). Let \( P \) be a cone and \( \theta \ll u \ll c \) for each \( c \in \text{int} \ P \), then \( u = \theta \).

**Lemma 2.11** ([21]). Let \( P \) be a cone. If \( u \in P \) and \( u \ll ku \) for some \( 0 \leq k < 1 \), then \( u = \theta \).

**Lemma 2.12** ([6]). Let \( P \) be a cone and \( a = b + c \) for each \( c \in \text{int} \ P \), then \( a \leq b \).

### 3. Main Results

In 2013, Huaping Huang and Shaoyuan Xu [7] proved the following theorem: Let \((X, d)\) be a complete cone b-metric space with the coefficient \( s \geq 1 \). Suppose the mapping \( T : X \times X \rightarrow X \) satisfies the condition \( d(Tx, Ty) \leq \alpha d(x, y) \), for all \( x, y \in X \). Where \( \alpha \in (0, 1) \) is a constant. Then \( T \) has a unique fixed point in \( X \). Furthermore, the iterative sequence \( \{T^n x\} \) converges to the fixed point. Now we expand this theorem in cone b-metric space as.
Theorem 3.1. Let \((X,d)\) be a complete cone b-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T : X \times X \to X\) satisfies the contractive condition.

\[
d(Tx,Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx) \tag{1}
\]

where the constant \(\alpha_i \in [0, 1); i = 1, 2, 3, 4, 5\) and \(\alpha_1 + \alpha_2 + \alpha_3 + s (\alpha_4 + \alpha_5) < \min\{1, \frac{2}{s}\}\). Then \(T\) has a unique fixed point in \(X\). Moreover, the iterative sequence \(\{T^n x\}\) converges to the fixed point.

Proof. Fix \(x_0 \in X\) and set \(x_1 = Tx_0\) and \(x_{n+1} = Tx_n = T^{n+1} x_0\). Firstly, we see

\[
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})
\]

\[
d(x_{n+1}, x_n) \leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, Tx_n) + \alpha_3 d(x_{n-1}, Tx_{n-1}) + \alpha_4 d(x_n, Tx_{n-1}) + \alpha_5 d(x_{n-1}, Tx_n)
\]

\[
\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + \alpha_4 d(x_n, x_n) + \alpha_5 d(x_{n-1}, x_{n+1})
\]

\[
\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + s \alpha_5 d(x_{n-1}, x_n) + d(x_n, x_{n+1})
\]

\[
\leq \alpha_1 d(x_n, x_{n-1}) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + s \alpha_5 d(x_{n-1}, x_n) + s \alpha_5 d(x_{n-1}, x_n) + s \alpha_5 d(x_{n-1}, x_n)
\]

\[
\leq (\alpha_2 + s \alpha_5) d(x_{n+1}, x_n) + (\alpha_1 + \alpha_3 + s \alpha_5) d(x_n, x_{n-1})
\]

It follows that

\[
(1 - \alpha_2 - s \alpha_5) d(x_{n+1}, x_n) \leq (\alpha_1 + \alpha_3 + s \alpha_5) d(x_n, x_{n-1}) \tag{2}
\]

Secondly,

\[
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) = d(Tx_{n-1}, Tx_n)
\]

\[
d(x_{n+1}, x_n) \leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_n, Tx_n) + \alpha_4 d(x_{n-1}, Tx_{n-1}) + \alpha_5 d(x_n, Tx_n)
\]

\[
\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x_{n-1}, x_{n+1}) + \alpha_5 d(x_n, x_n)
\]

\[
\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + s \alpha_4 d(x_{n-1}, x_n) + d(x_n, x_{n+1})
\]

\[
\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + s \alpha_4 d(x_{n-1}, x_n) + s \alpha_4 d(x_{n-1}, x_n) + s \alpha_4 d(x_{n-1}, x_n)
\]

\[
= (\alpha_3 + s \alpha_4) d(x_{n+1}, x_n) + (\alpha_1 + \alpha_2 + s \alpha_4) d(x_n, x_{n-1})
\]

This establishes that

\[
(1 - \alpha_3 - s \alpha_4) d(x_{n+1}, x_n) \leq (\alpha_1 + \alpha_2 + s \alpha_4) d(x_n, x_{n-1}) \tag{3}
\]

Adding up (2) and (3)

\[
d(x_{n+1}, x_n) \leq \frac{2 \alpha_1 + \alpha_2 + s (\alpha_4 + \alpha_5) - \alpha_3}{2 - \alpha_2 - \alpha_3 - s (\alpha_4 + \alpha_5)} d(x_n, x_{n-1})
\]

Put \(\alpha = \frac{2 \alpha_1 + \alpha_2 + s (\alpha_4 + \alpha_5)}{2 - \alpha_2 - \alpha_3 - s (\alpha_4 + \alpha_5)}\) it is easy to see that \(0 \leq \alpha < 1\). Now, proceeding in the same manner up to \(n\) iterations, we have:

\[
d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^n d(x_1, x_0)\]

for \(n \geq 1\) and letting \(m \geq n\), we have:

\[
d(x_{n+m}, x_n) \leq s d(x_{n+m}, x_{n+m-1}) + d(x_{n+m-1}, x_n)
\]

\[
= s d(x_{n+m}, x_{n+m-1}) + s d(x_{n+m-1}, x_n)
\]

\[
\leq s d(x_{n+m}, x_{n+m-1}) + s^2 d(x_{n+m-1}, x_{n+m-2}) + d(x_{n+m-2}, x_n)
\]

\[
\leq s d(x_{n+m}, x_{n+m-1}) + s^2 d(x_{n+m-1}, x_{n+m-2}) + s^3 d(x_{n+m-2}, x_{n+m-3}) + d(x_{n+m-3}, x_n)
\]
Let \( \theta \ll c \) be given, notice that \( \frac{\sum_{n=1}^{m-1} d(x_n,x_0)}{s - \alpha} \to \theta \) as \( m \to \infty \) for any \( k \). Making full use of Lemma 2.8, we find \( n_0 \in N \) such that
\[
\frac{s \alpha^{n_0+1}}{s - \alpha} d(x_1,x_0) + s^{-1} \alpha^d(x_1,x_0) \ll c
\]
\[
d(x_n,x_m) \leq \frac{\sum_{n=1}^{m-1} d(x_n,x_0)}{s - \alpha} d(x_1,x_0) + s^{-1} \alpha^d(x_1,x_0) \ll c
\]
for all \( n > n_0 \) and any \( m \). So by Lemma 2.9. Therefore \( \{x_n\} \) is a Cauchy sequence in \( (X,d) \). Since \( (X,d) \) is a complete cone-b metric space, there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \) for some \( x^* \in X \). Now we prove that \( x^* \) is the unique fixed point of \( T \). Actually, on the one hand,
\[
d(Tx^*,x^*) \leq s[d(Tx^*,Tx_n) + d(Tx_n,x^*)]
\]
\[
\leq sd(Tx^*,Tx_n) + sd(Tx_n,x^*)
\]
\[
\leq sd(Tx^*,Tx_n) + sd(x_n+1,x^*)
\]
\[
\leq s(\alpha d(x^*,x_n) + \alpha_2 d(x^*,Tx_n) + \alpha_3 d(x_n,Tx_n) + \alpha_4 d(x^*,Tx_n) + \alpha_5 d(x_n,Tx_n)) + s(x_n+1,x^*)
\]
\[
\leq s(\alpha d(x^*,x_n) + \alpha_2 d(x^*,Tx_n) + \alpha_3 d(x_n,x_{n+1}) + \alpha_4 d(x^*,x_{n+1}) + \alpha_5 d(x_n,x_{n+1})) + s d(x_{n+1},x^*)
\]
\[
\leq s(\alpha d(x^*,x_n) + \alpha_2 d(x^*,Tx_n) + \alpha_3 d(x_n,x_{n+1}) + \alpha_4 d(x^*,x_{n+1}) + \alpha_5 d(x_n,x_{n+1})) + s d(x_{n+1},x^*)
\]
\[
= \alpha_1 d(x^*,x_n) + \alpha_2 d(x^*,Tx_n) + s^2 \alpha_3 d(x_n,x^*) + s^2 \alpha_3 d(x^*,x_{n+1}) + s \alpha_4 d(x^*,x_{n+1}) + s^2 \alpha_5 d(x_n,x^*)
\]
\[
+ s^2 \alpha_5 d(x^*,Tx_n) + s d(x_{n+1},x^*)
\]
\[
\leq (s \alpha_2 + s^2 \alpha_3) d(x^*,Tx_n) + (s \alpha_4 + s^2 \alpha_5) d(x_n,x^*) + (s \alpha_3 + s \alpha_4 + s) d(x^*,x_{n+1})
\]
Such implies that
\[
(1 - s \alpha_2 - s^2 \alpha_3) d(x^*,Tx_n) \leq (s \alpha_1 + s^2 \alpha_3 + s^2 \alpha_5) d(x_n,x^*) + (s \alpha_3 + s \alpha_4 + s) d(x^*,x_{n+1})
\]
\[
(4)
\]
On the other hand,
\[
d(x^*,Tx_n) \leq s[d(x^*,Tx_n) + d(Tx_n,Tx^*)]
\]
\[
\leq sd(x^*,x_{n+1}) + sd(Tx_n,Tx^*)
\]
\[
\leq sd(x^*,x_{n+1}) + s(\alpha d(x_n,x^*) + \alpha_2 d(x_n,Tx_n) + \alpha_3 d(x^*,Tx_n) + \alpha_4 d(x_n,Tx^*) + \alpha_5 d(x^*,Tx_n))
\]
\[
= sd(x^*,x_{n+1}) + s(\alpha d(x_n,x^*) + \alpha_2 d(x_n,x_{n+1}) + \alpha_3 d(x^*,Tx_n) + \alpha_4 d(x_n,Tx^*) + \alpha_5 d(x^*,x_{n+1}))
\]
\[
\leq sd(x^*,x_{n+1}) + s(\alpha d(x_n,x^*) + \alpha_2 d(x_n,x_{n+1}) + \alpha_3 d(x^*,x_{n+1}) + \alpha_4 d(x^*,Tx_n) + \alpha_5 d(x^*,x_{n+1}))
\]
\[
= \alpha_1 d(x^*,x_n) + \alpha_2 d(x^*,Tx_n) + s^2 \alpha_3 d(x_n,x^*) + s^2 \alpha_3 d(x^*,x_{n+1}) + s \alpha_4 d(x^*,x_{n+1}) + s^2 \alpha_5 d(x_n,x^*)
\]
\[
+ s^2 \alpha_5 d(x^*,Tx_n) + s d(x_{n+1},x^*)
\]
\[
d(x^*,Tx^*) \leq (s \alpha_3 + s^2 \alpha_4) d(x^*,Tx_n) + (s \alpha_1 + s^2 \alpha_2 + s^2 \alpha_4) d(x_n,x^*) + (s + s^2 \alpha_2 + s \alpha_5) d(x^*,x_{n+1})
\]
(1 - sα1 - α4s2) d(x*, Tx*) ≤ (sα1 + s2α2 + s2α4) d(xn, x*) + (s + s2α2 + sα5) d(x*, xn+1) \tag{5}

Adding (4) and (5), we get

\[ d(x^*, T x^*) \leq \frac{(2sα1 + s^2α2 + s^2α4 + s^2α5)}{2 - sα2 - sα3 - s^2α4 - s^2α5} d(x_n, x^*) + \frac{(2s + s^2α2 + s^2α4 + sα5)}{2 - sα2 - sα3 - s^2α4 - s^2α5} d(x^*, x_{n+1}) \]

Since \{x_n\} is a Cauchy sequence, there for every \(c \in \text{int} \ P\), we select an \(n_i \in \mathbb{N}\) for all \(n \geq n_i\). Such that

\[ d(x^*, x_{n+1}) \leq \frac{(2s + s^2α2 + s^2α4 + s^2α5)}{2 - sα2 - sα3 - s^2α4 - s^2α5} c \]

\[ d(x_n, x^*) \leq \frac{(2sα1 + s^2α2 + s^2α4 + s^2α5)}{2 - sα2 - sα3 - s^2α4 - s^2α5} c \]

This for any \(c \in \text{int} \ P \) \(d(x^*, Tx^*) \ll c \forall n \geq n_i\). By Lemma 2.10 that \(d(x^*, Tx^*) = 0\) i.e. \(x^*\) is fixed point of \(T\) i.e. \(x^* = Tx^*\). Finally, we show the uniqueness of the fixed point. Indeed if there is another fixed point \(y^*\) then.

\[ d(x^*, y^*) = d(Tx^*, Ty^*) \]

\[ \leq \alpha_1 d(x^*, y^*) + \alpha_2 d(x^*, Tx^*) + \alpha_3 d(y^*, Ty^*) + \alpha_4 d(x^*, Ty^*) + \alpha_5 d(y^*, Tx^*) \]

\[ \leq \alpha_1 d(x^*, y^*) + \alpha_4 d(x^*, Ty^*) + \alpha_5 d(y^*, Tx^*) \]

\[ \leq \alpha_1 d(x^*, y^*) + \alpha_4 d(x^*, y^*) + \alpha_5 d(y^*, x^*) \]

\[ \leq \{\alpha_1 + s(\alpha_4 + \alpha_5)\} d(x^*, y^*) \]

Owing to \(0 \leq \alpha_1 + s(\alpha_4 + \alpha_5) < 1\). We deduce from Lemma 2.11 that \(x^* = y^*\).

\[ \square \]

**Corollary 3.2.** Let \((X, d)\) be complete cone b-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T: X \to X\) satisfies the contractive condition.

\[ d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 [d(x, Ty) + d(y, Tx)] \quad \text{for} \quad x, y \in X. \]

Where the constants \(\alpha_i \in [0, 1)\) and \(\alpha_1 + \alpha_2 + \alpha_2 + 2s\alpha_4 < \min \{1, \frac{2}{3}\}, \quad i = 1, 2, 3, 4\). Then \(T\) has a unique fixed point in \(X\).

**Corollary 3.3.** Let \((X, d)\) be complete cone b-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T: X \to X\) satisfies the contractive condition.

\[ d(Tx, Ty) \leq \alpha_1 d(x, y) \quad \text{for} \quad x, y \in X. \]

Where \(\alpha_1 \in [0, 1)\) is a constant. Then \(T\) has an unique fixed point.

**Proof.** Taking \(\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0\) in (1) we get the required result.

\[ \square \]

**Corollary 3.4.** Let \((X, d)\) be complete cone b-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T: X \to X\) satisfies the contractive condition.

\[ d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty). \]

Where the constants \(\alpha_i \in [0, 1)\) and \(\alpha_1 + \alpha_2 + \alpha_3 < \min \{1, \frac{2}{3}\}, \quad i = 1, 2, 3\). Then \(T\) has a unique fixed point.

**Proof.** Taking \(\alpha_4 = \alpha_5 = 0\) in (1) we get the required result.

\[ \square \]
Corollary 3.5. Let \((X,d)\) be complete cone \(b\)-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T : X \to X\) satisfies the contractive condition.

\[
d(Tx,Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)].
\]

Where the constants \(\alpha_i \in [0, 1)\) and \(\alpha_1 + 2\alpha_2 < \min \left\{1, \frac{s}{2}\right\}; \ i = 1, 2,\). Then \(T\) has a unique fixed point.

**Proof.** Taking \(\alpha_4 = \alpha_5 = 0, \ \alpha_3 = \alpha_2\) in (1) we get the required result.

Corollary 3.6. Let \((X,d)\) be complete cone \(b\)-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T : X \to X\) satisfies the contractive condition.

\[
d(Tx,Ty) \leq \alpha_4 [d(y, Tx) + d(x, Ty)].
\]

Where the constants \(\alpha_4 \in [0, \frac{1}{2})\). Then \(T\) has a unique fixed point.

**Proof.** Taking \(\alpha_1 = \alpha_2 = \alpha_3 = 0, \ \alpha_4 = \alpha_5\) in (1) we get the required result.

Corollary 3.7. Let \((X,d)\) be complete cone \(b\)-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T : X \to X\) satisfies the contractive condition.

\[
d(Tx,Ty) \leq \alpha_2 [d(x, Tx) + d(y, Ty)].
\]

Where the constants \(\alpha_2 \in [0, \frac{1}{2})\). Then \(T\) has a unique fixed point.

**Proof.** Taking \(\alpha = \alpha_4 = \alpha_5 = 0, \ \alpha_3 = \alpha_2\) in (1) we get the required result.

Corollary 3.8. Let \((X,d)\) be complete cone \(b\)-metric space with the coefficient \(s \geq 1\). Suppose the mapping \(T : X \to X\) satisfies the contractive condition.

\[
d(Tx,Ty) \leq \alpha_1 d(x, y) + \alpha_2 [d(x, Tx) + d(y, Ty)] + \alpha_3 [d(x, Ty) + d(y, Tx)].
\]

Where the constants \(\alpha_i \in [0, 1)\) and \(\alpha_1 + 2(\alpha_2 + \alpha_3 s) < \min \left\{1, \frac{s}{2}\right\}, \ i = 1, 2, 3\). Then \(T\) has a unique fixed point.

References

Some Common Fixed Point Theorems of Contractive Mappings in Cone b-metric Spaces


A Note on Extremal Disconnectedness

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Abstract: In this paper, we show that in an extremally disconnected topological space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. A characterization of extremal disconnectedness in terms of open filters is also obtained.

MSC: 54G05.

Keywords: Semi-open set, extremally disconnected space, open filter, open ultrafilter.

1. Introduction

The class of extremely disconnected (e.d.) topological spaces forms an important part of the class of all topological spaces. Gleason in [3] has shown that extremely disconnected topological spaces are precisely the projective spaces in the category of compact topological spaces and continuous maps. There are many equivalent definitions of extremal disconnectedness ([4–6, 8, 14]). Most of these results involve either open sets or closed sets or both open and closed sets. Some weak forms of open sets (closed sets) like pre-open (pre-closed), semi-open (semi-closed) and $\alpha$-open ($\alpha$-closed) exist in literature. For details, the reader is referred to [1, 2, 9–13, 15]. Some equivalent forms of extremal disconnectedness are known in the form of families of sets, where the closure operator is distributive over the intersection of every two members of the family. In [8], the concept of “rounding” for any open filter on a topological space is introduced to characterize extremal disconnectedness.

In this paper, we show that closure operator is distributive over the intersection of every two semi-open sets of an e.d. topological space. In this situation, there arises a natural question of finding a family (possibly the largest) of subsets of an e.d. topological space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. It is shown that such largest family does not exist. However in an e.d. topological space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Finally, the necessary and sufficient conditions for a family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family to be largest, are obtained. A new characterization of extremal disconnectedness is also obtained using the concept of ‘rounding’ for any open filter.

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2. Notation and Definitions

For completeness, we have included some of the standard notation and definitions. Herein by a space we mean a topological space. Let X be a space. For \( A \subseteq X \), \( \text{int}_X(A) \) and \( \text{cl}_X(A) \) denote interior and closure of A respectively. A subset A of a space X is called semi-open ([9]) (pre-open ([2]) if \( A \subseteq \text{cl}_X(\text{int}_X(A)) \) (A \( \subseteq \text{int}_X(\text{cl}_X(A)) \)). A set whose complement is semi-open (pre-open) is called semi-closed (pre-closed). An open filter \( F \) on X is a prime open filter if for open sets \( A, B \) of X, \( A \cup B \in F \) only if \( A \in F \) or \( B \in F \). An open ultrafilter on X is a maximal open filter. For an open filter \( F \) on X, the \( \{G \subseteq X : G \text{ is open in } X \text{ and } x \in X\} \). odX is used to denote the open filter \( \{G \subseteq X : G \text{ is open in } X \text{ and } cl_X(G) = X\} \).

Following [12], a subset A of a space X is said to be n-regularly nowhere dense (for n an integer greater than 1) if there exists open sets \( A_1, A_2, \ldots, A_n \) in X such that \( A \subseteq \cap \{cl_X(A_i) : i = 1, 2, \ldots, n\} \) and \( \cap \{A_i : i = 1, 2, \ldots, n\} = \emptyset \). Let R denotes the open filter \( \{X - A : A \text{ is n-regularly nowhere dense for some } n > 1\} \).

3. Extremal Disconnectedness and Semi-open Sets

We shall take following as the definition of an extremely disconnected space.

**Definition 3.1.** A space X is called an extremally disconnected (e.d.) space ([7]) if closure of every open set of X is open in X.

**Lemma 3.2.** Let X be an e.d. space. Let A and B be two semi-open sets in X. Then \( cl_X(A \cap B) = cl_X(A) \cap cl_X(B) \).

**Proof.** As A and B are semi-open sets in X, \( cl_X(A) \cap cl_X(B) = cl_X(int_X(A)) \cap cl_X(int_X(B)) \). Since X is an e.d., \( cl_X(int_X(A)) \cap cl_X(int_X(B)) = cl_X(int_X(A) \cap int_X(B)) \). This implies that \( cl_X(A \cap B) = cl_X(A) \cap cl_X(B) \).

**Lemma 3.3.** Let \( \varsigma \) be a family of subsets of a space X containing the family of all open sets of X, where the closure operator is distributive over the intersection of every two members of \( \varsigma \). Then for every \( A \in \varsigma \), \( cl_X(A) \) is open in X.

**Proof.** Let \( A \in \varsigma \), \( X - cl_X(A) \in \varsigma \) by the given condition. \( cl_X(A) \cap cl_X(X - cl_X(A)) = cl_X(A \cap (X - cl_X(A))) = cl_X(\emptyset) = \emptyset \) again by the given condition. This implies that \( cl_X(A) \subseteq int_X(cl_X(A)) \); so \( cl_X(A) \) is open. Hence the result follows.

**Lemma 3.4.** Let X be an e.d. space. Let A be semi-open and B pre-open in X. \( cl_X(A \cap B) = cl_X(A) \cap cl_X(B) \).

**Proof.** As A is semi-open and B pre-open in X, \( cl_X(A) \cap cl_X(B) = cl_X(int_X(A)) \cap cl_X(int_X(cl_X(B))) \). Since X is an e.d., \( cl_X(int_X(A)) \cap cl_X(int_X(cl_X(B))) = cl_X(int_X(A) \cap int_X(cl_X(B))) \). This implies that \( cl_X(A) \cap cl_X(B) = cl_X(A \cap B) \).

**Remark 3.5.** It is known that in an e.d. space, the family of all open sets is a family where closure operator is distributive over the intersection of every two members of the family ([8]). Lemma 3.2 makes that family considerably larger by replacing open sets by semi-open sets as every open set is semi-open. Since each \( \alpha \)-open set is semi-open and every regular closed set is semi-open, the family of semi-open sets become quite a large family. But still there is a scope to determine a larger family (possibly the largest) of subsets of an e.d. space where the closure operator is distributive over the intersection of every two members of the family. The answer to this question is not in affirmative. Consider following example in support of this.

Let \( IN \) be the space of all natural numbers with cofinite topology. Let A be the set of all even natural numbers. Then \( cl_X(A) = cl_X(IN - A) = IN \). So A and \( IN - A \) are pre-open sets of IN. Let \( \delta \) denotes the family of all semi-open sets of IN.
Let $\varsigma$ be the largest family of subsets of the space containing $\delta$, where the closure operator is distributive over the intersection of every two members of $\varsigma$. Now, using Lemma 3.4, $\{A\} \cup \delta$ is a family of subsets of $X$ containing $\delta$ where the closure operator is distributive over the intersection of every two members of this family. So $\{A\} \cup \delta \subset \varsigma$. Similarly $\{IN-A\} \cup \delta \subset \varsigma$. Therefore $cl_X(A \cap (IN-A)) = cl_X(A) \cap cl_X(IN-A)$. This is not possible as $cl_X(A \cap (IN-A)) = \phi$ and $cl_X(A) \cap cl_X(IN-A) = IN$.

Hence there does not exist largest family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of any two members of the family.

As justified in Remark 3.5, in general, there does not exist the largest family of subsets of an e.d. space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. So one can think of finding a maximal such family. The following lemma is a step to show the existence of such a maximal family.

**Lemma 3.6.** Let $X$ be a space. Let $\varsigma$ be a family of subsets of a space $X$ containing the family of all semi-open sets, where the closure operator is distributive over the intersection of any two members of $\varsigma$. Then there exists a maximal family of subsets of the space containing $\varsigma$ where the closure operator is distributive over the intersection of every two members of $\varsigma$.

**Proof.** The union of a chain of families of subsets of a space where the closure operator is distributive over the intersection of every two members of the family, becomes a family where the closure operator is distributive over the intersection of every two members of the family. Now the existence of a maximal such family follows using Zorn’s Lemma.

**Theorem 3.7.** In an e.d. space, there exists a maximal family of subsets of the space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Such a family has to be a subfamily of the family of all pre-open sets.

**Proof.** The proof follows using Lemma 3.2, Remark 3.5 and Lemma 3.6.

Theorem 3.7 assures the existence of a maximal family of subsets of an e.d. space containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family. Though in general such a largest family does not exist (see Remark 3.5). There may be some spaces where the existence of such largest family of subsets is possible. The following theorem gives necessary and sufficient conditions for the existence of such family of subsets of a space. The proof is left for the readers.

**Theorem 3.8.** The following are equivalent for a space.

(1). The space has a largest family of subsets containing the family of all semi-open sets, where the closure operator is distributive over the intersection of every two members of the family.

(2). Closure operator is distributive over the intersection of every two members of the family of all pre-open sets of the family.

4. Extremal Disconnectedness and Open Filters

First we note the following proposition.

**Proposition 4.1.** Let $X$ be a space. Then $rR = odX$.

**Theorem 4.2.** Let $X$ be a space. If $F$ is a prime open filter containing $R$, then $rF$ is an open ultrafilter.

**Proof.** Using Proposition 3.3 of [12], there exists an open ultrafilter $F^*$ on $X$ such that $\{int_X(cl_X(A)) : A \in F^*\} \subset F$. By Proposition 2.3 (k)(2) of [15], $F$ is contained in a unique open ultrafilter. Therefore $rF$ is contained in a unique open
A Note on Extremal Disconnectedness

 ultrafilter using Proposition 3(e) of [8]. As \( rF \) is round by Proposition 3(a) of [8], so \( rF \) is an open ultrafilter by Proposition 3(f) of [8].

**Theorem 4.3.** Let \( X \) be a space. If \( R \) is a prime open filter, then \( odX \) is the only open ultrafilter.

**Proof.** By Theorem 4.2, \( rR \) is an open ultrafilter. So \( odX \) is an open ultrafilter by Proposition 4.1. Now the proposition follows by Remark 4(d) of [8].

**Remark 4.4.** Let \( X \) be a space. If \( R \) is a prime open filter, then by Theorem 4.3, \( odX \) is the only open ultrafilter. This implies that every non-empty open subset of \( X \) is dense in \( X \). Therefore \( X \) is an e.d. space.

**Theorem 4.5.** The following are equivalent for a space \( X \).

(1). \( X \) is extremally disconnected.

(2). Every prime open filter contains \( R \).

**Proof.** We only prove (2)\( \Rightarrow \)(1). Suppose (2) holds. Let \( x \in X \). Since \( ox \) is a prime open filter containing \( R \), so by Theorem 4.2, \( rox \) is an open ultrafilter. Now (1) follows by Proposition 4 of [8].

**References**


An Extension of Common Fixed Point Theorem in D-Metric Space

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Abstract: There have been a number of generalizations of Metric spaces. D–Metric space is one such generalization initiated by Dhage [1] in 1984 and open new research area. Then many authors have obtained interesting fixed point results in D–Metric space satisfying contractive type condition. In this paper we proved some fixed point theorems in D–Metric space and also proved new fixed point theorem D–Metric space for a contractive self–maps.

MSC: 47H10.

Keywords: D-Metric space, fixed point theorems.

2. Some Preliminaries and Definitions

Definition 2.1 ([7]). Let $X$ be a non–empty set. Let function $d : X \times X \times X \rightarrow [0, \infty)$ is called a D–Metric if $d$ satisfies,

(D1) $d(x, y, z) = 0$ iff $x = y = z$ (coincidence)

(D2) $d(x, y, z) = d(p(x, y, z))$ (Symmetry). Where $p$ is a permutation of $x, y, z.$

(D3) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ (Tetrahedral inequality)

The non–empty set $X$ together with D–Metric “$d$” is called D–Metric space and it is denoted by $(X, d).$
Example 2.2. Define $d : R^n \times R^n \times R^n \to [0, \infty)$ by, $d(a, b, c) = \alpha \max\{||a - b||, ||b - c||, ||c - a||\}$ for all $a, b, c \in R^n$, $\alpha > 0$. Where $|| \|$ is norm in Euclidean space $R^n$, is a D-Metric on $R^n$. Hence $(R^n, d)$ is a D-Metric space.

Definition 2.3 ([2]). Sequence $\{a_n\}$ in D-Metric space is called D-Cauchy if $\lim_{m,n,p} d(a_n, a_m, a_p) = 0$.

Definition 2.4 ([2]). Sequence $\{a_n\}$ in D-Metric space $(X, d)$ is said to be D-Convergent to $a \in X$ if $\lim_{m,n} d(a_n, a_m, a) = 0$. That is, for a point $a \in X$, if each $\epsilon > 0$ there exist positive integer $n_0$ such that $d(a_n, a_m, a) < \epsilon$ for all $n, m > n_0$.

Definition 2.5 ([3]). Let $(X, d)$ be D-Metric space. A subset $U$ of $X$ is said to be bounded if these exist constant $s > 0$ such that, $d(a, b, c) < s$, for all $a, b, c \in U$ and $s$ is called D-bound of $U$. For a bounded sequence $\{y_n\}$ in D-Metric space $(X, d)$, let $a_n = \delta(\{y_n, y_{n+1}, y_{n+2}, \ldots\})$ for $n \in N$. Then $a_n$ is finite for all $n \in N$ and $\{a_n\}$ is non increasing and $a_n > 0$ for all $n \in N$ so there is an $a > 0$ such that, $\lim_{n \to \infty} a_n = a$.

Definition 2.6 ([2]). Every D-Cauchy sequence converges to a point in D-Metric space is called complete D-Metric space.

Definition 2.7 ([6]). Consider $(X, d)$ be D-Metric space and $f : X \to X$. The orbit of $f$ at the point $a \in X$ is the set $O(a) = \{a, fa, f^2a, \ldots\}$.

Definition 2.8 ([6]). Consider $(X, d)$ be D-Metric space and $O(a)$ be orbit of $f : X \to X$ is said to be bounded if there exists a constant $C > 0$ such that $d(x, y, z) \leq C$ for all $x, y, z \in O(a)$. The constant $C$ is called D-bound of $O(a)$. D-Metric space is said to be $f$-orbitally bounded if $O(a)$ is bounded for each $a \in X$.

Definition 2.9 ([5]). An orbit $O(a)$ is said to be $f$-orbitally complete if every D-Cauchy sequence in $O(a)$ converges to a point in $X$.

Definition 2.10 ([5]). Let $f : X \to X$ is called $\alpha$-condensing if, for any bounded set $B \subseteq X$, $f(B)$ is bounded and $\alpha(f(B)) < \alpha(B)$ if $\alpha(B) > 0$. Many authors refer $\alpha$-condensing maps as densifying mapping.

3. Some Fixed Point Theorems on D-Metric Space

Theorem 3.1 ([6]). Let $f$ be a self map on $X$ and $f$-orbitally bounded, $X$ be complete D-Metric space also $\alpha$-condensing. Then $0(a)$ is compact for each $a \in X$.

Proof. Consider $a \in X$ and define $B \subseteq X$ by $B = \{a_n\}$ where $a_n = f^n a$ then, $B = \{a, fa, f^2a, \ldots\} = \{a\} \cup \{fa, f^2a, \ldots\} = \{a\} \cup f(B)$.  \hspace{1cm} (1)

Therefore if $B$ is not precompact, then $\alpha(B) = \alpha(f(B)) < \alpha(B)$ which is contradiction. Therefore $B = O(a)$ is compact.  \hspace{1cm} \Box

Theorem 3.2. Let $f : X \to X$, where $f$ is continuous compact of a bounded D-Metric space such that,

$$d(f^m a, f^m b, f^m c) < \delta(a, b, c) \text{ for } a, b, c \in X$$ \hspace{1cm} (2)

with two of $\{a, b, c\}$ are distinct and $\delta(a, b, c)$ defined as $\delta(a, b, c) = \delta(O(a) \cup O(b) \cup O(c))$ also $x, y, z$ are fixed positive integer. Then $f$ has a unique fixed point.
Proof. Since $f$ is compact, there is a compact subset $A$ of $X$ containing $f(X)$. Then $f(A) \subset A$ and $B = \bigcap_{n=1}^{\infty} f^n(A) \neq \emptyset$ be compact $f$-invariant subset of $X$. Since $B$ is compact there is $a, b, c \in B$ such that, $\delta(B) = d(a, b, c)$ and let $\delta(B) > 0$.

Since $f(B) = B$, there is $a_1, b_1, c_1 \in B$ such that $a = f_{a_1}$, $b = f_{b_1}$ and $c = f_{c_1}$, form (2)

$$\delta(B) = d(a, b, c) = d(f_{a_1}, f_{b_1}, f_{c_1}) < \delta(a, b, c) = \delta(B) \quad (3)$$

Which is a contradiction, therefore $B$ contain single point, which is fixed point of $f$. Now, to show uniqueness, suppose $s$ and $t$ are fixed point of $f$ such that $s \neq t$ Then from (2)

$$0 < d(s, s, t) = d(f_{a_1}, f_{b_1}, f_{c_1}) < d(s, s, t) \quad (4)$$

Which is a contradiction. Therefore there is a unique fixed point.

Corollary 3.3 ([7]). Let $f$ be a continuous self map on $X$, where $X$ is compact $D$–Metric space satisfying,

$$d(f_{a}, f_{b}, f_{c}) < \max\{d(a, b, c), d(a, f_{a}, c), d(b, f_{b}, c), d(a, f_{a}, c), d(b, f_{a}, c), d(q, q, p)\} \quad (5)$$

For all $a, b, c \in X$ with $a \neq f_{a}$, $b \neq f_{b}$, or $c \neq f_{c}$. Then $f$ has a unique fixed point $q$ in $X$.

Proof. From (5) we get $d(f_{a}, f_{b}, f_{c}) < \delta(a, b, c)$ and from Theorem 3.2 we get the existence and uniqueness of a fixed point $q$.\hfill $\Box$

Theorem 3.4 ([4]). Let $\{b_n\}$ be a bounded sequence in $D$–Metric space $X$ with $\ell$ as $D$–bound such that,

$$d(b_n, b_{n+1}, b_m) \leq \emptyset^n(\ell) \quad (6)$$

For all $m > n \in N$, where $\emptyset : R^+ \rightarrow R^+$ satisfies $\sum_{n=1}^{\infty} \emptyset^n(s) < \infty$ for each $s \in R^+$. Then $\{b_n\}$ is $D$–Cauchy.

Let $\psi$ denote the class of all real functions $\emptyset : R^+ \rightarrow R^+$ satisfying

1. $\emptyset$ is continuous
2. $\emptyset$ is increasing
3. $\emptyset(s) < s$ if $s > 0$ and
4. $\sum_{n=1}^{\infty} \emptyset^n(s) < \infty$ for $s \in R^+$.

The element of the class $\psi$ is called control function and commonly used control function as $\emptyset(s) = ks$, $0 \leq k < 1$. Now using Definition 2.5, 2.8 and 2.9 we generalized Theorem 3.2 and Corollary 3.3 as follows.

4. Main Result

Theorem 4.1. Consider $f, g$ be self mapping on $X$ such that,

$$d(f_{a}, f_{b}, f_{c}) \leq \alpha \emptyset(\max\{d(ga, gb, gc) + d(gb, fa, gc), d(gb, fs, gc) + d(ga, fb, gc), d(ga, fa, gc) + d(ga, gb, gc)\}) \quad (7)$$

for $a, b, c \in X$ with $0 \leq \alpha < \frac{1}{2}$ and $\emptyset \in \psi$. Also,
1. \( f(X) \subseteq g(X) \)

2. \( g(X) \) is complete and \( f(X) \) is bounded.

3. \( g \) and \( f \) are commuting.

Then \( g \) and \( f \) have unique common fixed point \( t \in X \).

**Proof.** Consider \( a_0 = a \in X \) and define \( \{b_n\} \) in \( X \) by

\[
b_0 = ga_0, \quad b_{n+1} = fa_n = ga_{n+1}, n = 0, 1, 2, \ldots
\]

(8)

because \( g(X) \supseteq f(X) \). If \( b_r = b_{r+1} \) for \( r \in N \) then, \( b_r = f_{r+1} = f_{r+1} = ga_r = b_{r+1} = t \), for some \( t \in X \). Now, we prove that, \( t \) is a common fixed point of \( g \) and \( f \). Because \( f_{r+1} = ga_r \) and \( g, f \) are coincidentally commuting. Therefore \( ft = fg \) and \( ft = gt \). Now,

\[
d(ft, gt, t) = d(ft, fga, fa_r) = d(ft, ft, fa_r) \\
\leq \alpha \theta \left( \max \{d(gt, gt, ga_r) + d(gt, ft, ga_r), d(gt, ft, ga_r) + d(gt, ft, ga_r) + d(gt, gt, ga_r) \} \right) \\
\leq \alpha \theta \left( \max \{d(gt, ft, t) + d(gt, ft, t) + d(gt, ft, t) + d(gt, ft, t) + d(gt, ft, t) \} \right) \\
\leq 2\alpha \theta(d(gt, ft, t)) \\
\leq \theta(d(ft, gt, t))
\]

Gives \( ft = gt = t \). Therefore \( t \) is a common fixed point of \( g \) and \( f \). Therefore assume that, \( b_n \neq b_{n+1} \) for all \( n \) in \( N \). Now, we prove that, \( \{b_n\} \) is \( D \)-Cauchy. For \( m > 1 \),

\[
d(b_1, b_2, b_m) = d(fa_1, fa_2, fa_{m-1}) \\
\leq \alpha \theta \left( \max \{d(ga_1, ga_2, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) \} \right) \\
\leq \alpha \theta \left( \max \{d(b_1, b_2, b_{m-1}) + d(b_1, b_2, b_{m-1}) + d(b_1, b_2, b_{m-1}) + d(b_1, b_2, b_{m-1}) \} \right) \\
\leq 2\alpha \theta \left( \max \{d(b_1, b_2, b_2) \} \right) \leq 2\alpha \theta(k) \leq \theta(k)
\]

For \( m > 2 \),

\[
d(b_2, b_3, b_m) = d(fa_1, fa_2, fa_{m-1}) \\
\leq \alpha \theta \left( \max \{d(ga_1, ga_2, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) + d(ga_1, fa_1, ga_{m-1}) + d(ga_1, ga_2, ga_{m-1}) \} \right) \\
\leq \alpha \theta \left( \max \{d(b_2, b_3, b_{m-1}) + d(b_2, b_3, b_{m-1}) + d(b_2, b_3, b_{m-1}) + d(b_1, b_2, b_{m-1}) \} \right) \\
\leq 2\alpha \theta \left( \max \{d(b_2, b_3, b_2) \} \right) \leq 2\alpha \theta^2(k) \leq \theta^2(k)
\]
In general for $m > n$,
\[
d(b_n, b_{n+1}, b_m) \leq 2\alpha^{n-1} \left( \max_{i,j} d(b_i, b_{j}, b_{m-n}) \right), \quad \text{where } 0 \leq i \leq n \text{ and } 1 \leq j \leq n + 1.
\]
\[
\leq 2\alpha^n(k) \leq \emptyset^0(k)
\]

By Theorem 3.4 gives $\{b_n\}$ is D-Cauchy. Since $g(X)$ is complete there exist point $t \in g(X)$ such that, $\lim b_n = t$ i.e., $\lim_{n \to \infty} b_n = \lim_{n \to \infty} g a_n = t$. Now to prove that $t$ is a common fixed point of $g$ and $f$. Because $t \in g(X)$ there is a point $q \in X$ such that $gq = t$. Now, we show that, $fq = gq = t$.

\[
d(fq, gq, gq) = \lim_n d(fq, fn, fn)
\]
\[
\leq \lim_n \alpha(t) \max \{d(gq, ga_n, ga_n) + d(ga_n, fn, fn), d(ga_n, fn, fn)
\]
\[
+ d(gq, fn, fn), d(gq, fn, fn) + d(gq, ga_n, ga_n)\}
\]
\[
\leq \alpha(t) \max\{0 + d(t, fn, t), 0 + 0, d(t, fn, t) + 0\}
\]
\[
\leq \alpha(t) \max\{d(t, fn, t)\} \leq \emptyset(d(t, fn, t))
\]

Which gives $fq = t$ since $\emptyset \in \psi$ then $t = fn = gq$ is a common fixed point of $g$ and $f$. To prove uniqueness, let $s(\neq t)$ be common fixed point of $f$ and $g$ then,

\[
d(t, t, s) = d(ft, ft, fs)
\]
\[
\leq \alpha(t) \max\{d(ft, ft, gs) + d(gt, ft, gs), d(gt, ft, gs) + d(gt, ft, gs), d(gt, ft, gs) + d(gt, gt, gs)\}
\]
\[
\leq \alpha(t) \max\{d(t, t, s) + d(t, t, s) + d(t, s, s) + d(t, t, s) + d(t, t, s)\}
\]
\[
\leq 2\alpha(t) \max\{d(t, t, s), d(t, s, s)\}
\]
\[
\leq 2\alpha(t) \emptyset(d(t, t, s)) < \emptyset(d(t, t, s))
\]

Because $d(t, t, s) < \emptyset(d(t, t, s))$ is not possible. Therefore

\[
d(t, t, s) \leq \emptyset(d(t, s, s))
\]

(9)

Now, interchanging the role of $t$ and $s$ we get,

\[
d(t, s, s) \leq \emptyset(d(t, t, s))
\]

(10)

From (9) and (10) we get,

\[
d(t, t, s) \leq \emptyset^2(d(t, t, s))
\]

which is contradiction. Therefore $t = s$.

\section{Conclusion}

In this paper we generalized fixed point theorem in D-Metric space and find out fixed point by using some contractive conditions.
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References

Contra Semi*δ-Continuous Functions in Topological Spaces

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Abstract: In this paper we define contra-semi*δ-continuous, contra-semi*δ-irresolute, semi*δ-open and semi*δ-closed functions and investigate their properties.

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1. Introduction

In 1996, Dontchev introduced and investigated the notions of contra-continuity. Later Zainab Aodia Athbanaih introduced and investigated the concept of contra (δ, gδ)-continuous functions. Quite recently the authors have introduced the concept of semi*δ-open sets and studied their properties. The aim of this paper is to introduce and investigate a new class of functions called contra-semi*δ-continuous, contra-semi*δ-irresolute, semi*δ-open and semi*δ-closed.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X, τ), cl(A) and int(A) denote the closure and the interior of A respectively. We recall some known definitions needed in this paper.

Definition 2.1. Let (X, τ) be a topological space. A subset A of the space X is said to be

(1). Semi-open if A ⊆ Cl(int(A)) and semi*-open if A ⊆ Cl*(int(A)).

(2). Pre open if A ⊆ Int(cl(A)) and pre*-open if A ⊆ Int*(cl(A)).

(3). Semi-pre open if A ⊆ Cl(int(cl(A))) and semi*-pre open if A ⊆ Cl*(pInt(A))).

(4). α-open if A ⊆ Int(Cl(int(A))) and α*-open if A ⊆ Int*(Cl(int*(A))).
(5). Regular-open if \( A = \text{Int} (\text{Cl}(A)) \) and \( \delta \)-open if \( A = \delta \text{Int}(A) \).

(6). \( \alpha \)-open if \( A \subseteq \text{Cl}(\alpha \text{Int}(A)) \) and \( \text{semi}^*\alpha \)-open if \( A \subseteq \text{Cl}^*(\alpha \text{Int}(A)) \).

(7). \( \delta \)-semi-open if \( A \subseteq \text{Cl} (\delta \text{Int}(A)) \) and \( \text{semi}^*\delta \)-open \( A \subseteq \text{Cl}^*(\delta \text{Int}(A)) \).

The complements of the above mentioned sets are called their respective closed sets.

**Definition 2.2.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be

(1). contra-continuous \([4]\) if \( f^{-1}(V) \) is closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(2). contra-g-continuous \([2]\) if \( f^{-1}(V) \) is g-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(3). contra-semi-continuous \([5]\) if \( f^{-1}(V) \) is semi-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(4). contra-semi*-continuous \([10]\) if \( f^{-1}(V) \) is semi*-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(5). contra-pre-continuous \([7]\) if \( f^{-1}(V) \) is pre-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(6). contra-\( \alpha \)-continuous \([6]\) if \( f^{-1}(V) \) is \( \alpha \)-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(7). contra-\( \alpha^* \)-continuous \([9]\) if \( f^{-1}(V) \) is \( \alpha^* \)-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(8). contra-semi-pre-continuous \([3]\) if \( f^{-1}(V) \) is semi-pre closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(9). contra-semi*pre-continuous \([8]\) if \( f^{-1}(V) \) is semi*-pre closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(10). contra-semia-continuous \([15]\) if \( f^{-1}(V) \) is semi-\( \alpha \)-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(11). contra-semi*\( \alpha \)-continuous \([15]\) if \( f^{-1}(V) \) is semi*\( \alpha \)-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

(12). contra-\( \delta \)-continuous \([16]\) if \( f^{-1}(V) \) is \( \delta \)-closed in \((X, \tau)\) for every open set \( V \) in \((Y, \sigma)\).

**Theorem 2.3** \([14]\). Every \( \delta \)-open set is open.

**Theorem 2.4** \([11, 12]\). Every \( \delta \)-open set is semi*\( \delta \)-open and every \( \delta \)-closed set is semi*\( \delta \)-closed.

**Theorem 2.5** \([11]\). In any topological space,

(1). Every semi*\( \delta \)-open set is \( \delta \)-semi-open.

(2). Every semi*\( \delta \)-open set is semi-open.

(3). Every semi*\( \delta \)-open set is semi*-open.

(4). Every semi*\( \delta \)-open set is semi*-preopen.

(5). Every semi*\( \delta \)-open set is semi-preopen.

(6). Every semi*\( \delta \)-open set is semi\( \alpha \)-open

(7). Every semi*\( \delta \)-open set is semi\( \alpha \)-closed.

**Remark 2.6** \([12]\). Similar results for semi*\( \delta \)-closed sets are also true.

**Theorem 2.7** \([11]\). Arbitrary union of semi*\( \delta \)-open sets in \( X \) is also semi*\( \delta \)-open in \( X \).
Theorem 2.8 ([11]). For a subset $A$ of a topological space $(X, \tau)$ the following statements are equivalent:

1. $A$ is semi*-δ-open.
2. $A \subseteq Cl^*(\delta Int(A))$.
3. $Cl^*(\delta Int(A)) = Cl^*(A)$.

Theorem 2.9 ([12]). For a subset $A$ of a topological space $(X, \tau)$, the following statements are equivalent:

1. $A$ is semi*-δ-closed.
2. $Int^*(\delta Cl(A)) \subseteq A$.
3. $Int^*(\delta Cl(A)) = Int^*(A)$.

Theorem 2.10. A subset $A$ of a space $X$ is

1. semi*-δ-open if and only if $s^*\delta Int(A) = A$ [11].
2. semi*-δ-closed if and only if $s^*\delta Cl(A) = A$ [12].

3. Contra-Semi*-δ-Continuous Functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called contra-semi*-δ-continuous if $f^{-1}(V)$ is semi*-δ-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.

Example 3.2. Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. $S^*\delta C(X, \tau) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = d$, $f(b) = c$, $f(c) = a$, $f(d) = b$. Clearly, $f$ is contra-semi*-δ-continuous.

Theorem 3.3. Every contra-δ-continuous function is contra-semi*-δ-continuous.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be contra-δ-continuous. Let $V$ be an open set in $(Y, \sigma)$. Since $f$ is contra-δ-continuous, $f^{-1}(V)$ is δ-closed in $(X, \tau)$. By Theorem 2.4, $f^{-1}(V)$ is semi*-δ-closed in $(X, \tau)$. Hence $f$ is contra-semi*-δ-continuous.

Remark 3.4. It can be seen that the converse of the above theorem is not true.

Definition 3.5. A function $f : (X, \tau) \to (Y, \sigma)$ is called contra-semi*-δ-continuous if $f^{-1}(V)$ is δ-semi-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.

Theorem 3.6. In any topological space,

1. Every contra-semi*-δ-continuous function is contra-δ-semi-continuous.
2. Every contra-semi*-δ-continuous function is contra-semi-continuous.
3. Every contra-semi*-δ-continuous function is contra-semi*-continuous.
4. Every contra-semi*-δ-continuous function is contra-semi*-pre-continuous.
5. Every contra-semi*-δ-continuous function is contra-semi-pre-continuous.
6. Every contra-semi*-δ-continuous function is contra-semi*-α-continuous.
(7). Every contra-semi*-δ-continuous function is contra-semi-α-continuous.

Proof.

(1). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is δ-semi-closed in \((X, \tau)\). Hence \( f \) is contra-δ-semi-continuous.

(2). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-α-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-α-continuous.

(3). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-pre-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-pre-continuous.

(4). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-pre-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-pre-continuous.

(5). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-pre-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-pre-continuous.

(6). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-α-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-α-continuous.

(7). Let \( f : (X, \tau) \to (Y, \sigma) \) be contra-semi*-δ-continuous. Let \( V \) be an open set in \((Y, \sigma)\). Since \( f \) is contra-semi*-δ-continuous, \( f^{-1}(V) \) is semi*-δ-closed in \((X, \tau)\). By Remark 2.6, \( f^{-1}(V) \) is semi-α-closed in \((X, \tau)\). Hence \( f \) is contra-semi*-α-continuous.

Remark 3.7. The converse of each of the statements in Theorem 3.6 is not true.

Remark 3.8. The concepts of contra-semi*-δ-continuous and contra-continuous (resp. contra-g-continuous, contra-α-continuous, contra-pre-continuous, contra-α*-continuous, contra-pre*-continuous) are independent.

Remark 3.9. The composition of two contra-semi*-δ-continuous functions need not be contra-semi*-δ-continuous and this can be shown by the following example.

Example 3.10. Let \( X = Y = Z = \{a, b, c, d\} \), \( \tau = \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X \), \( \sigma = \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, Y \) and \( \eta = \emptyset, \{a\}, \{b\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = d; f(b) = c; f(c) = b; f(d) = a \) and define \( g : (Y, \sigma) \to (Z, \eta) \) by \( g(a) = c; g(b) = g(c) = g(d) = a \). Then \( f \) and \( g \) are contra-semi*-δ-continuous but \( g \circ f \) is not semi*-δ-continuous. Since \( \{a\} \) is open in \((Z, \eta)\) but \((g \circ f)^{-1}(\{a\}) = f^{-1}(g^{-1}(\{a\})) = f^{-1}(\{b, c, d\}) = \{a, b, c\} \) which is not semi*-δ-closed in \((X, \tau)\).

Theorem 3.11. For a function \( f : (X, \tau) \to (Y, \sigma) \), the following are equivalent:

(1). \( f \) is contra-semi*-δ-continuous.

(2). For each \( x \in X \) and each closed set \( F \) in \( Y \) containing \( f(x) \), there exists a semi*-δ-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq F \).
(3) The inverse image of each closed set in $Y$ is semi*-δ-open in $X$.

(4) $\text{Cl}(\delta \text{Int}(f^{-1}(F))) = \text{Cl}^*(f^{-1}(F))$ for every closed set $F$ in $Y$.

(5) $\text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V))$ for every open set $V$ in $Y$.

**Proof.**

(1)$\Rightarrow$(2) Let $f : (X,\tau) \to (Y,\sigma)$ be contra-semi*-δ-continuous. Let $x \in X$ and $F$ be a closed set in $Y$ containing $f(x)$. Then $V = Y \setminus F$ is an open set in $Y$ not containing $f(x)$. Since $f$ is contra-semi*-δ-continuous, $f^{-1}(V)$ is semi*-δ-closed in $X$ not containing $x$. That is, $f^{-1}(V) = X \setminus f^{-1}(F)$ is a semi*-δ-closed set in $X$ not containing $x$. Therefore $U = f^{-1}(F)$ is a semi*-δ-open set in $X$ containing $x$ such that $f(U) \subseteq F$.

(2)$\Rightarrow$(3) Let $F$ be a closed set in $Y$. Let $x \in f^{-1}(F)$, then $f(x) \in F$. By (2), there is a semi*-δ-open set $U_x$ in $X$ containing $x$ such that $f(x) \in f(U_x) \subseteq F$. That is, $x \in U_x \subseteq f^{-1}(F)$. Therefore $f^{-1}(F) = \bigcup\{U_x : x \in f^{-1}(F)\}$. By Theorem 2.7, $f^{-1}(F)$ is semi*-δ-open in $X$.

(3)$\Rightarrow$(4) Let $F$ be a closed set in $Y$. By (3), $f^{-1}(F)$ is a semi*-δ-open set in $X$. By Theorem 2.8, $\text{Cl}^*(\delta \text{Int}(f^{-1}(F))) = \text{Cl}^*(f^{-1}(F))$.

(4)$\Rightarrow$(5) If $V$ is any open set in $Y$, then $Y \setminus V$ is closed in $Y$. By (4), we have $\text{Cl}^*(\delta \text{Int}(f^{-1}(Y \setminus V))) = \text{Cl}^*(f^{-1}(Y \setminus V))$. Taking the complements, we get $\text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V))$.

(5)$\Rightarrow$(1) Let $V$ be any open set in $Y$. Then by assumption, $\text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V))$. By Theorem 2.9, $f^{-1}(V)$ is semi*-δ-closed.

**Theorem 3.12.** If $f : (X,\tau) \to (Y,\sigma)$ is semi*-δ-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is contra-continuous, then $g \circ f : (X,\tau) \to (Z,\eta)$ is contra-semi*-δ-continuous.

**Proof.** Let $V$ be an open set in $(Z,\eta)$. Since $g$ is contra-continuous, $g^{-1}(V)$ is closed in $(Y,\sigma)$. Since $f$ is semi*-δ-continuous, and hence by Theorem 3.36 [13], $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*-δ-closed in $(X,\tau)$. Hence $g \circ f$ is contra-semi*-δ-continuous.

**Theorem 3.13.** If $f : (X,\tau) \to (Y,\sigma)$ is contra-semi*-δ-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is continuous, then $g \circ f : (X,\tau) \to (Z,\eta)$ is contra-semi*-δ-continuous.

**Proof.** Let $V$ be an open set in $(Z,\eta)$. Since $g$ is continuous, $g^{-1}(V)$ is open in $(Y,\sigma)$. Since $f$ is contra-semi*-δ-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*-δ-closed in $(X,\tau)$. Hence $g \circ f$ is contra-semi*-δ-continuous.

**Theorem 3.14.** If $f : (X,\tau) \to (Y,\sigma)$ is contra-semi*-δ-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is contra-continuous, then $g \circ f : (X,\tau) \to (Z,\eta)$ is semi*-δ-continuous.

**Proof.** Let $V$ be an open set in $(Z,\eta)$. Since $g$ is contra-continuous, $g^{-1}(V)$ is closed in $(Y,\sigma)$. Since $f$ is contra-semi*-δ-continuous, and hence by Theorem 3.11 $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*-δ-open in $(X,\tau)$. Therefore, $g \circ f$ is semi*-δ-continuous.

**Theorem 3.15.** If $f : (X,\tau) \to (Y,\sigma)$ is semi*-δ-irresolute and $g : (Y,\sigma) \to (Z,\eta)$ is contra semi*-δ-continuous, then their composition $g \circ f : (X,\tau) \to (Z,\eta)$ is contra semi*-δ-continuous.

**Proof.** Let $V$ be an open set in $(Z,\eta)$. Since $g$ is contra semi*-δ-continuous, then $g^{-1}(V)$ is semi*-δ-closed in $(Y,\sigma)$ and since $f$ is semi*-δ-irresolute, by invoking Theorem 4.5 [13], $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*-δ-closed in $(X,\tau)$. Therefore, $g \circ f$ is contra semi*-δ-continuous.
Theorem 3.16. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra-$\delta$-continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is contra-semi*$\delta$-continuous.

Proof. Let $V$ be an open set in $(Z, \eta)$. Since, $g$ is continuous, then $g^{-1}(V)$ is open in $(Y, \sigma)$ and since $f$ is contra-$\delta$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\delta$-closed in $(X, \tau)$. Hence by Theorem 2.4, $(g \circ f)^{-1}(V)$ is semi*$\delta$-closed in $(X, \tau)$. Therefore, $g \circ f$ is contra-semi*$\delta$-continuous.

Theorem 3.17. If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra semi*$\delta$-continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $\delta$-continuous, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is contra-semi*$\delta$-continuous.

Proof. Let $V$ be an open set in $(Z, \eta)$. Since, $g$ is $\delta$-continuous, then $g^{-1}(V)$ is $\delta$-open in $(Y, \sigma)$ and by Theorem 2.3 $g^{-1}(V)$ is open in $(Y, \sigma)$. Since $f$ is contra-semi*$\delta$-continuous, then $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*$\delta$-closed in $(X, \tau)$. Therefore, $g \circ f$ is contra-semi*$\delta$-continuous.

Theorem 3.18. Let $X, Y$ be any topological spaces and $Y$ be $T_{1/2}$-space. Then the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ of contra-semi*$\delta$-continuous function $f : (X, \tau) \rightarrow (Y, \sigma)$ and the $\sigma$-continuous function $g : (Y, \sigma) \rightarrow (Z, \eta)$ is contra-semi*$\delta$-continuous.

Proof. Let $V$ be an closed set in $(Z, \eta)$. Since $g$ is $\sigma$-continuous, then $g^{-1}(V)$ is $\sigma$-closed in $(Y, \sigma)$ and $Y$ is $T_{1/2}$-space, $g^{-1}(V)$ is closed in $(Y, \sigma)$. Since $f$ is contra-semi*$\delta$-continuous, by Theorem 3.11 $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is semi*$\delta$-open in $(X, \tau)$. Therefore, again by Theorem 3.11 $g \circ f$ is contra-semi*$\delta$-continuous.

Definition 3.19. A topological space $(X, \tau)$ is said to be $T_{31/2}$-space, if every semi*$\delta$-open set of $X$ is open in $X$.

Theorem 3.20. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a contra-semi*$\delta$-continuous function and $X$ be a $T_{31/2}$-space. Then $f$ is contra-continuous.

Proof. Let $V$ be any closed set in $(Y, \sigma)$. Since $f$ is contra semi*$\delta$-continuous, $f^{-1}(V)$ is semi*$\delta$-open in $(X, \tau)$. Then by assumption, $f^{-1}(V)$ is open in $(X, \tau)$. Therefore, $f$ is contra-continuous.

Theorem 3.21. If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is semi*$\delta$-continuous and if $Y$ is locally indiscrete, then $f$ is contra-semi*$\delta$-continuous.

Proof. Let $V$ be an open set in $(Y, \sigma)$. Since $Y$ is locally discrete, $V$ is closed in $(Y, \sigma)$. Since, $f$ is semi*$\delta$-continuous, $f^{-1}(V)$ is semi*$\delta$-closed in $(X, \tau)$. Therefore, $f$ is contra-semi*$\delta$-continuous.

Theorem 3.22. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function and $g : X \rightarrow X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then $f$ is contra-semi*$\delta$-continuous if $g$ is contra-semi*$\delta$-continuous.

Proof. Let $V$ be an open subset of $(Y, \sigma)$. Then $X \times V$ is an open subset of $X \times Y$. Since $g$ is a contra-semi*$\delta$-continuous, then $g^{-1}(X \times V)$ is semi*$\delta$-closed subset of $X$. Also, $g^{-1}(X \times V) = f^{-1}(V)$. Hence, $f$ is contra-semi*$\delta$-continuous.

4. Contra Semi*$\delta$-Irresolute Functions

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be contra-semi*$\delta$-irresolute if $f^{-1}(V)$ is semi*$\delta$-closed in $(X, \tau)$ for every semi*$\delta$-open set $V$ in $(Y, \sigma)$.

Theorem 4.2. For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent:
(1). \( f \) is contra-semi\( \delta \)-irresolute.

(2). For each \( x \in X \) and each semi\( \delta \)-closed set \( F \) in \( Y \) with \( f(x) \in F \), there exists a semi\( \delta \)-open set \( U \) in \( X \) such that \( x \in U \) and \( f(U) \subseteq F \).

(3). The inverse image of each semi\( \delta \)-closed set in \( Y \) is semi\( \delta \)-open in \( X \).

(4). \( Cl^*(\delta \text{Int}(f^{-1}(F))) = Cl^*(f^{-1}(F)) \) for every semi\( \delta \)-closed set \( F \) in \( Y \).

(5). \( \text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V)) \) for every semi\( \delta \)-open set \( V \) in \( Y \).

Proof.

(1)\( \Rightarrow \) (2) Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be contra-semi\( \delta \)-irresolute. Let \( x \in X \) and \( F \) be a semi\( \delta \)-closed set in \( Y \) containing \( f(x) \). Then \( V = Y \setminus F \) is semi\( \delta \)-open set in \( Y \) not containing \( f(x) \). Since \( f \) is contra-semi\( \delta \)-irresolute, \( f^{-1}(V) \) is semi\( \delta \)-closed set in \( X \) not containing \( x \). That is, \( f^{-1}(V) = f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \) is a semi\( \delta \)-closed set in \( X \) not containing \( x \). Therefore \( U = f^{-1}(F) \) is a semi\( \delta \)-open set in \( X \) containing \( x \) such that \( f(U) \subseteq F \).

(2)\( \Rightarrow \) (3) Let \( F \) be a semi\( \delta \)-closed set in \( Y \). Let \( x \in f^{-1}(F) \), then \( f(x) \in F \). By (2), there exists a semi\( \delta \)-open set \( U_x \) in \( X \) containing \( x \) such that \( f(x) \in f(U_x) \subseteq F \). That is, \( x \in U_x \subseteq f^{-1}(F) \). Therefore \( f^{-1}(F) = \cup \{ U_x : x \in f^{-1}(F) \} \).

By Theorem 2.7, \( f^{-1}(F) \) is semi\( \delta \)-open in \( X \).

(3)\( \Rightarrow \) (4) Let \( F \) be a semi\( \delta \)-closed set in \( Y \). By (3), \( f^{-1}(F) \) is a semi\( \delta \)-open set in \( X \). By Theorem 2.8, \( Cl^*(\delta \text{Int}(f^{-1}(F))) = Cl^*(f^{-1}(F)) \).

(4)\( \Rightarrow \) (5) If \( V \) is any semi\( \delta \)-open set in \( Y \), then \( Y \setminus V \) is semi\( \delta \)-closed in \( Y \). By (4), we have \( Cl^*(\delta \text{Int}(f^{-1}(Y \setminus V))) = Cl^*(f^{-1}(Y \setminus V)) \). Taking the complements, we get \( \text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V)) \).

(5)\( \Rightarrow \) (1) Let \( V \) be any semi\( \delta \)-open set in \( Y \). Then by (5), \( \text{Int}^*(\delta \text{Cl}(f^{-1}(V))) = \text{Int}^*(f^{-1}(V)) \). By Theorem 2.9, \( f^{-1}(V) \) is semi\( \delta \)-closed. Therefore \( f \) is contra-semi\( \delta \)-irresolute.

\[ \square \]

Theorem 4.3. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be semi\( \delta \)-irresolute and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be contra-semi\( \delta \)-irresolute. Then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is contra-semi\( \delta \)-irresolute.

Proof. Let \( V \) be a semi\( \delta \)-open set in \( (Z, \eta) \). Since \( g \) is contra-semi\( \delta \)-irresolute, \( g^{-1}(V) \) is semi\( \delta \)-closed in \( (Y, \sigma) \). Since \( f \) is semi\( \delta \)-irresolute, by invoking Theorem 4.5 [13], \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is semi\( \delta \)-closed in \( (X, \tau) \). Hence \( g \circ f \) is contra-semi\( \delta \)-irresolute.

\[ \square \]

Theorem 4.4. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be contra-semi\( \delta \)-irresolute and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be semi\( \delta \)-irresolute. Then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is contra-semi\( \delta \)-irresolute.

Proof. Let \( V \) be a semi\( \delta \)-open set in \( (Z, \eta) \). Since \( g \) is semi\( \delta \)-irresolute, \( g^{-1}(V) \) is semi\( \delta \)-closed in \( (Y, \sigma) \). Since \( f \) is contra-semi\( \delta \)-irresolute, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is semi\( \delta \)-closed in \( (X, \tau) \). Hence \( g \circ f \) is contra-semi\( \delta \)-irresolute.

\[ \square \]

Theorem 4.5. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be contra-semi\( \delta \)-irresolute and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be contra-semi\( \delta \)-irresolute. Then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is semi\( \delta \)-irresolute.

Proof. Let \( V \) be a semi\( \delta \)-open set in \( (Z, \eta) \). Since \( g \) is contra-semi\( \delta \)-irresolute, \( g^{-1}(V) \) is semi\( \delta \)-closed in \( (Y, \sigma) \). Since \( f \) is contra-semi\( \delta \)-irresolute, by Theorem 4.2, \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is semi\( \delta \)-open in \( (X, \tau) \). Hence \( g \circ f \) is semi\( \delta \)-irresolute.

\[ \square \]
5. Open and Closed Functions Associated with Semi*δ-Open Sets

Definition 5.1. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be semi*δ-open if \( f(V) \) is semi*δ-open in \( Y \) for every open set \( V \) in \( X \).

Definition 5.2. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be contra-semi*δ-open if \( f(V) \) is semi*δ-closed in \( Y \) for every open set \( V \) in \( X \).

Definition 5.3. A function \( f : X \rightarrow Y \) is said to be pre-semi*δ-open if \( f(V) \) is semi*δ-open in \( Y \) for every semi*δ-open set \( V \) in \( X \).

Definition 5.4. A function \( f : X \rightarrow Y \) is said to be contra-pre-semi*δ-open if \( f(V) \) is semi*δ-closed in \( Y \) for every semi*δ-open set \( V \) in \( X \).

Definition 5.5. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be semi*δ-closed if \( f(F) \) is semi*δ-closed in \( Y \) for every closed set \( F \) in \( X \).

Definition 5.6. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be contra-semi*δ-closed if \( f(F) \) is semi*δ-open in \( Y \) for every closed set \( F \) in \( X \).

Definition 5.7. A function \( f : X \rightarrow Y \) is said to be pre-semi*δ-closed if \( f(F) \) is semi*δ-closed in \( Y \) for every semi*δ-closed set \( F \) in \( X \).

Definition 5.8. A function \( f : X \rightarrow Y \) is said to be contra-pre-semi*δ-closed if \( f(F) \) is semi*δ-open in \( Y \) for every semi*δ-closed set \( F \) in \( X \).

Remark 5.9. The composition of two semi*δ-closed maps need not be semi*δ-closed in general as shown in the following example.

Example 5.10. Consider \( X = Y = Z = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \), \( \sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c, Y\} \) and \( \eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, b, c\}, \{a, b, c, d\}, Z\} \). \( S^*\delta C(Y, \sigma) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}, \{a, b, c, d, f\}, \{a, b, c, d, e, f\}, \{a, b, c, d, e, f, g\}, Z\} \) and \( \eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}, \{a, b, c, d, e, f\}, \{a, b, c, d, e, f, g\}, Z\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = b, f(b) = a, f(c) = d, f(d) = a \). Clearly, \( f \) is semi*δ-closed. Consider the map \( g : (Y, \sigma) \rightarrow (Z, \eta) \) defined by \( g(a) = g(b) = d, g(c) = a, g(d) = c \), clearly \( g \) is semi*δ-closed. But \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is not semi*δ-closed. Since \( g \circ f(\{a, d\}) = g(f(\{a, d\})) = g(\{a, b\}) = \{d\} \) which is not semi*δ-closed in \( (Z, \eta) \).

Theorem 5.11. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a bijection. Then the following are equivalent:

1. \( f^{-1} \) is semi*δ-continuous.
2. \( f \) is semi*δ-open.
3. \( f \) is semi*δ-closed.

Proof.

1)\( \Rightarrow \)2) Let \( V \) be an open set in \( X \). Since, \( f^{-1} \) is semi*δ-continuous map then \( (f^{-1})^{-1}(V) = f(V) \) is semi*δ-open set in \( Y \). Hence, \( f \) is semi*δ-open map.

2)\( \Rightarrow \)3) Let \( F \) be a closed set of \( X \). Then \( X\setminus F \) is an open set in \( X \). By hypothesis \( f(X\setminus F) \) is semi*δ-open in \( Y \). Since, \( f(X\setminus F) = Y\setminus f(F) \). Hence, \( f(F) \) is semi*δ-closed in \( Y \). Therefore, \( f \) is semi*δ-closed.
(3)⇒(1) Let \( F \) be a closed set in \( X \). Then \( f(F) \) is semi\(^*\)\(\delta\)-closed in \( Y \). Since \( (f^{-1})^{-1}(F) = f(F) \), which is semi\(^*\)\(\delta\)-closed set in \( Y \). Therefore, \( f \) is semi\(^*\)\(\delta\)-continuous. 

\( \square \)

**Theorem 5.12.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be surjective.

1. If \( (g \circ f) \) is semi\(^*\)\(\delta\)-open and \( f \) is continuous, then \( g \) is semi\(^*\)\(\delta\)-open.

2. If \( (g \circ f) \) is continuous and \( f \) is semi\(^*\)\(\delta\)-open, then \( g \) is semi\(^*\)\(\delta\)-continuous.

3. If \( (g \circ f) \) is semi\(^*\)\(\delta\)-continuous and \( g \) is an open map, then \( f \) is semi\(^*\)\(\delta\)-continuous.

4. If \( (g \circ f) \) is open and \( g \) is semi\(^*\)\(\delta\)-continuous, then \( f \) is semi\(^*\)\(\delta\)-open.

5. If \( (g \circ f) \) is semi\(^*\)\(\delta\)-open, \( f \) is semi\(^*\)\(\delta\)-continuous, and \( (X, \tau) \) is a \( T_{\delta-} \)-space, then \( g \) is semi\(^*\)\(\delta\)-open.

**Proof.**

1. Let \( V \) be an open set in \( (Y, \sigma) \). Then, \( f^{-1}(V) \) is an open set in \( (X, \tau) \). Since \( (g \circ f) \) is semi\(^*\)\(\delta\)-open, \( (g \circ f)(f^{-1}(V)) = g(V) \) is semi\(^*\)\(\delta\)-open in \( (Z, \eta) \). Therefore, \( g \) is semi\(^*\)\(\delta\)-open.

2. Let \( V \) be any open set in \( (Z, \eta) \). Since \( (g \circ f) \) is continuous \( (g \circ f)^{-1}(V) \) is open in \( (X, \tau) \). Since \( f \) is semi\(^*\)\(\delta\)-open \( f((g \circ f)^{-1}(V)) = f^{-1}(g^{-1}(V)) = f^{-1}(V) \) which is semi\(^*\)\(\delta\)-open in \( (Y, \sigma) \). Therefore \( f \) is semi\(^*\)\(\delta\)-continuous.

3. Let \( V \) be any open set in \( (Y, \sigma) \). Since \( g \) is open, \( g(V) \) is open in \( (Z, \eta) \). Also since \( (g \circ f) \) is semi\(^*\)\(\delta\)-continuous \( (g \circ f)^{-1}(g(V)) = f^{-1}(g^{-1}(g(V))) = f^{-1}(V) \) is semi\(^*\)\(\delta\)-open in \( (X, \tau) \). Therefore \( f \) is semi\(^*\)\(\delta\)-continuous.

4. Let \( V \) be an open set in \( (X, \tau) \). Since \( (g \circ f) \) is an open map, \( (g \circ f)(V) \) is an open set in \( (Z, \eta) \). Also since \( g \) is semi\(^*\)\(\delta\)-continuous, \( g^{-1}(g \circ f)(V) = f(V) \) is semi\(^*\)\(\delta\)-open in \( (Y, \sigma) \). Hence, \( f \) is semi\(^*\)\(\delta\)-open.

5. Let \( V \) be a open set in \( (Y, \sigma) \). Since \( f \) is semi\(^*\)\(\delta\)-continuous, \( f^{-1}(V) \) is semi\(^*\)\(\delta\)-open in \( (X, \tau) \). Since \( X \) is a \( T_{\delta-} \)-space, \( f^{-1}(V) \) is open in \( (X, \tau) \). Since \( g \circ f \) is semi\(^*\)\(\delta\)-open and \( f \) is surjective, \( (g \circ f)(f^{-1}(V)) = g(V) \) is semi\(^*\)\(\delta\)-open in \( (Z, \eta) \).

Thus \( g \) is semi\(^*\)\(\delta\)-open map. \( \square \)

**Theorem 5.13.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent:

1. \( f \) is semi\(^*\)\(\delta\)-open.

2. \( f(\text{Int}(A)) \subseteq s^*\text{Int}(f(A)) \) for every subset \( A \) of \( X \).

**Proof.**

(1)⇒(2) Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be semi\(^*\)\(\delta\)-open. Let \( A \) be a subset of \( X \). Then \( \text{Int}(A) \) is an open set in \( X \). Since \( f \) is semi\(^*\)\(\delta\)-open, \( f(\text{Int}(A)) \) is semi\(^*\)\(\delta\)-open set in \( Y \). We have \( \text{Int}(A) \subseteq (A) \). Thus \( f(\text{Int}(A)) \subseteq f(A) \). Then \( s^*\text{Int}(f(\text{Int}(A))) \subseteq s^*\text{Int}(f(A)) \) which implies \( f(\text{Int}(A)) \subseteq s^*\text{Int}(f(A)) \).

(2)⇒(1) Let \( A \) be any open set in \( X \). Then \( \text{Int}(A) = A \). Thus \( f(\text{Int}(A)) = f(A) \). But \( f(\text{Int}(A)) \subseteq s^*\text{Int}(f(A)) \). That is \( f(A) \subseteq s^*\text{Int}(f(A)) \). Also \( s^*\text{Int}(f(A)) \subseteq f(A) \). By Theorem 2.10 (1), \( f(A) \) is semi\(^*\)\(\delta\)-open and hence \( f \) is semi\(^*\)\(\delta\)-open. \( \square \)

**Theorem 5.14.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent:

1. \( f \) is semi\(^*\)\(\delta\)-closed.

2. \( s^*\text{Cl}(f(A)) \subseteq f(\text{Cl}(A)) \) for every subset \( A \) of \( X \).
Proof.

(1)⇒(2) Let \(f : (X, \tau) \to (Y, \sigma)\) be semi*δ-closed. Let \(A\) be a subset of \(X\). Then \(cl(A)\) is a closed set in \(X\). Since \(f\) is a semi*δ-closed map, \(f(cl(A))\) is a semi*δ-closed set in \(Y\). We have \(A \subseteq cl(A)\). Thus \(f(A) \subseteq f(cl(A))\). Then \(s^*\delta cl(f(A)) \subseteq s^*\delta cl(f(cl(A))) = f(cl(A))\).

(2)⇒(1) Let \(A\) be any closed set in \(X\). Then \(A = cl(A)\). Thus \(f(A) = f(cl(A))\). But \(s^*\delta cl(f(A)) \subseteq f(cl(A)) = f(A)\). Also \(f(A) \subseteq s^*\delta cl(f(A))\). By Theorem 2.10 (2), \(f(A)\) is semi*δ-closed and hence \(f\) is semi*δ-closed.

**Theorem 5.15.** A map \(f : (X, \tau) \to (Y, \sigma)\) is semi*δ-open if and only if for each subset \(S\) of \((Y, \sigma)\) and for each closed set \(F\) of \((X, \tau)\) containing \(f^{-1}(S)\), there exists a semi*δ-closed set \(V\) of \((Y, \sigma)\) such that \(S \subseteq V\) and \(f^{-1}(V) \subseteq F\).

**Proof.** Suppose that \(f\) is semi*δ-open. Let \(S \subseteq Y\) and \(F\) be a closed set of \((X, \tau)\) such that \(f^{-1}(S) \subseteq F\). Now \(X \setminus F\) is an open set in \((X, \tau)\). Since \(f\) is semi*δ-open map, \(f(X \setminus F)\) is semi*δ-open set in \((Y, \sigma)\). Then, \(V = Y \setminus f(X \setminus F)\) is a semi*δ-closed set in \((Y, \sigma)\). Note that \(f^{-1}(S) \subseteq F\) implies \(S \subseteq V\) and \(f^{-1}(V) = X \setminus f^{-1}(X \setminus F) = F\). That is, \(f^{-1}(V) \subseteq F\).

Conversely, let \(B\) be an open set of \((X, \tau)\). Then, \(f^{-1}((f(B))^c) \subseteq B^c\) and \(B^c\) is a closed set in \((X, \tau)\). By hypothesis, there exists a semi*δ-closed set \(V\) of \((Y, \sigma)\) such that \((f(B))^c \subseteq V\) and \(f^{-1}(V) \subseteq B^c\) and so \(B \subseteq (f^{-1}(V))^c\). Hence \(V^c \subseteq f(B) \subseteq ((f^{-1}(V))^c)\) which implies \(f(B) = V^c\). Since \(V^c\) is a semi*δ-open. \(f(B)\) is semi*δ-open in \((Y, \sigma)\) and therefore \(f\) is semi*δ-open.

**Theorem 5.16.** If \(f : (X, \tau) \to (Y, \sigma)\) is closed map and \(g : (Y, \sigma) \to (Z, \eta)\) is semi*δ-closed, then the composition \(g \circ f : (X, \tau) \to (Z, \eta)\) is semi*δ-closed map.

**Proof.** Let \(V\) be any closed set in \((X, \tau)\). Since \(f\) is closed map, \(f(V)\) is a closed set in \((Y, \sigma)\). Since \(g\) is semi*δ-closed map, \((g(f(V)))\) is semi*δ-closed in \((Z, \eta)\) which implies \((g \circ f)(V) = g(f(V))\) is semi*δ-closed in \((Z, \eta)\) and hence \(g \circ f\) is semi*δ-closed.

**Remark 5.17.** If \(f : (X, \tau) \to (Y, \sigma)\) is semi*δ-closed map and \(g : (Y, \sigma) \to (Z, \eta)\) is closed, then the composition \(g \circ f : (X, \tau) \to (Z, \eta)\) is not semi*δ-closed map as shown in the following example.

**Example 5.18.** Consider \(X = Y = Z = \{a, b, c, d\}\), \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}\) and \(\sigma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, Y\}\), \(\eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, Z\}\), \(S^*\delta C(Y, \sigma) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, Y\}\), \(S^*\delta C(Z, \eta) = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{b, c\}, \{b, c, d\}, \{b, d\}, \{a, b, c, d\}\}\). Let \(f : (X, \tau) \to (Y, \sigma)\) be defined by \(f(a) = b, f(b) = a, f(c) = d, f(d) = a\). Clearly, \(f\) is semi*δ-closed. Consider the map \(g : (Y, \sigma) \to (Z, \eta)\) defined by \(g(a) = c, g(b) = d, g(c) = a, g(d) = d\), clearly \(g\) is closed. But \(g \circ f : (X, \tau) \to (Z, \eta)\) is not semi*δ-closed, since \(g \circ f\{\{c, d\}\} = g\{f\{c, d\}\} = g\{a, d\} = \{c, d\}\) which is not semi*δ-closed in \((Z, \eta)\).

**Theorem 5.19.** If \(f : (X, \tau) \to (Y, \sigma)\) is g-closed, \(g : (Y, \sigma) \to (Z, \eta)\) is semi*δ-closed and \((Y, \sigma)\) is \(T_{\frac{1}{2}}\) space, then the composition \(g \circ f : (X, \tau) \to (Z, \eta)\) is semi*δ-closed.

**Proof.** Let \(V\) be any closed set in \((X, \tau)\). Since \(f\) is g-closed, \(f(V)\) is g-closed in \((Y, \sigma)\) and since \(Y\) is \(T_{\frac{1}{2}}\) space, \(f(V)\) is closed in \((Y, \sigma)\). Also \(g\) is semi*δ-closed, \(g(f(V)) = (g \circ f)(V)\) is semi*δ-closed in \((Z, \eta)\). Therefore, \(g \circ f\) is semi*δ-closed.

**Theorem 5.20.** Let \(f : (X, \tau) \to (Y, \sigma)\) and \(g : (Y, \sigma) \to (Z, \eta)\) be two mappings such that their composition \(g \circ f : (X, \tau) \to (Z, \eta)\) be semi*δ-closed mapping. Then the following statements are true.

(1) If \(f\) is continuous and surjective, then \(g\) is semi*δ-closed.
(2). If $g$ is semi*-δ-irresolute and injective, then $f$ is semi*-δ-closed.

(3). If $f$ is $g$-continuous, surjective and $(X, \tau)$ is a $T_1^2$ space, then $g$ is semi*-δ-closed.

Proof.

(1). Let $V$ be a closed set in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(V)$ is closed in $(X, \tau)$. Also since $g \circ f$ is semi*-δ-closed which implies $(g \circ f)(f^{-1}(V))$ is semi*-δ-closed in $(Z, \circ)$. That is $g(V)$ is semi*-δ-closed in $(Z, \circ)$, since $f$ is surjective. Therefore, $g$ is semi*-δ-closed.

(2). Let $V$ be a closed set in $(X, \tau)$. Since $g \circ f$ is semi*-δ-closed, $(g \circ f)(V)$ is semi*-δ-closed in $(Z, \circ)$. Also since $g$ is semi*-δ-irresolute, $g^{-1}(g \circ f(V))$ is semi*-δ-closed in $(Y, \sigma)$. That is $f(V)$ is semi*-δ-closed in $(Y, \sigma)$, since $g$ is injective. Therefore, $f$ is semi*-δ-closed.

(3). Let $V$ be a closed set in $(Y, \sigma)$. Since $f$ is $g$-continuous, $f^{-1}(V)$ is g-closed in $(X, \tau)$ and $X$ is a $T_2$ space, $f^{-1}(V)$ is closed in $(X, \tau)$. Since $g$?f is semi*-δ-closed, $(g \circ f)(f^{-1}(V))$ is semi*-δ-closed in $(Z, \eta)$. That is $g(V)$ is semi*-δ-closed in $(Z, \eta)$, since $f$ is surjective. Therefore, $g$ is semi*-δ-closed.

Theorem 5.21. Let $f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \eta)$ be semi*-δ-open maps and $(Y, \sigma)$ be $T_{S*\delta}$-space. Then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is semi*-δ-open.

Proof. Let $V$ be an open set in $(X, \tau)$. By assumption $f(V)$ is semi*-δ-open in $(Y, \sigma)$. Since $Y$ is a $T_{S*\delta}$-space, $f(V)$ is open in $(Y, \sigma)$ and again by assumption $g(f(V))$ is semi*-δ-open in $(Z, \eta)$. Thus $g \circ f(V)$ is semi*-δ-open in $(Z, \eta)$. Hence $g \circ f$ is semi*-δ-open.

Theorem 5.22. Let $f : X \to Y$ and be $g : Y \to Z$ be functions.

(1). $g \circ f$ is pre-semi*-δ-open if both $f$ and $g$ are pre-semi*-δ-open.

(2). $g \circ f$ is semi*-δ-open if $f$ is semi*-δ-open and $g$ is pre-semi*-δ-open.

(3). $g \circ f$ is pre-semi*-δ-closed if both $f$ and $g$ are pre-semi*-δ-closed.

(4). $g \circ f$ is semi*-δ-closed if $f$ is semi*-δ-closed and $g$ is pre-semi*-δ-closed.

Proof.

(1). Let $V$ be any semi*-δ-open set in $(X, \tau)$. Since $f$ is pre-semi*-δ-open, $f(V)$ is semi*-δ-open set in $(Y, \sigma)$. Also since $g$ is pre-semi*-δ-open, $g(f(V)) = (g \circ f)(V)$ is semi*-δ-open set in $(Z, \eta)$. Hence $g \circ f$ is pre-semi*-δ-open.

(2). Let $V$ be any open set in $(X, \tau)$. Since $f$ is semi*-δ-open, $f(V)$ is semi*-δ-open set in $(Y, \sigma)$. Also since $g$ is pre-semi*-δ-open, $g(f(V)) = (g \circ f)(V)$ is semi*-δ-open set in $(Z, \eta)$. Hence $g \circ f$ is semi*-δ-open.

(3). Let $F$ be any semi*-δ-closed set in $(X, \tau)$. Since $f$ is pre-semi*-δ-closed, $f(F)$ is semi*-δ-closed set in $(Y, \sigma)$. Also since $g$ is pre-semi*-δ-closed, $g(f(F)) = (g \circ f)(F)$ is semi*-δ-closed set in $(Z, \eta)$. Hence $g \circ f$ is pre-semi*-δ-closed.

(4). Let $F$ be any closed set in $(X, \tau)$. Since $f$ is semi*-δ-closed, $f(F)$ is semi*-δ-closed set in $(Y, \sigma)$. Also since $g$ is pre-semi*-δ-closed, $g(f(F)) = (g \circ f)(F)$ is semi*-δ-closed set in $(Z, \eta)$. Hence $g \circ f$ is semi*-δ-closed.
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References

[16] Zainab Aodia Athbanaih, On Contra ($\delta$, g$\delta$)-Continuous Functions, University of Al-Qadissiya, College of Education.
On a Simple Inequality Problem

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Abstract: This note introduces five different proofs of a simple inequality problem applying five well-known inequalities.

MSC: 26D15

Keywords: AM-GM Inequality, Radon’s Inequality, Rearrangement Inequality.

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1. Introduction

Mathematical Inequality is a useful and powerful tool in many mathematical areas. In mathematical contests we can also find many problems requiring proving inequalities. However, this topic is almost untouched in intro Math courses, and, in most cases, only slightly mentioned in advanced Math courses. As a result, students sometimes use lots complicated calculation to prove or calculate a fairly simple problem, due to their lack of these knowledge. In Cvetkovski’s book [2], he provided the following example.

Example 1.1. Prove that for every positive real number \( a, b, c \) we have

\[
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c.
\]

This seemingly easy problem can be proved in many ways. However, if one plans to prove it without using any already known inequalities, it becomes extremely difficult. Using this as an example, I will in this note first introduce five well-known inequalities. I will then use them one by one and provide five different proofs.

2. Proofs of the Problem

All the inequalities introduced in this section, together with their proofs, can be found in many inequality focused textbooks, like [4, 5], and the classic book [3] by Hardy, Littlewood, and Pólya. Here I refer to [2] as my main source of notations.

**Theorem 2.1** (AM-GM Inequality). Let \( a_1, a_2, \ldots, a_n \) be positive real numbers. The numbers \( AM = \frac{a_1 + a_2 + \cdots + a_n}{n} \) and \( GM = \sqrt[n]{a_1 a_2 \cdots a_n} \) are called the arithmetic mean and geometric mean for the numbers \( a_1, a_2, \ldots, a_n \), respectively, and we have \( AM \geq GM \). Equality occurs if and only if \( a_1 = a_2 = \cdots = a_n \).

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Using this inequality, we may find the first proof of Example 1.1. This proof was provided in [2] at p.16.

Proof. Applying AM-GM inequality, we have

$$\frac{a^2}{b} + b \geq 2a, \quad \frac{b^2}{c} + c \geq 2b, \quad \frac{c^2}{a} + a \geq 2c.$$ 

Summing them together we have \(a^2 b + b^2 c + c^2 a + (a + b + c) \geq 2(a + b + c)\), which can be easily simplified to the claimed inequality in Example 1.1.

**Theorem 2.2** (Rearrangement Inequality). Let \(a_1 \leq a_2 \leq \cdots \leq a_n\) and \(b_1 \leq b_2 \leq \cdots \leq b_n\) be real numbers. For any permutation \((x_1, x_2, \cdots, x_n)\) of \((a_1, a_2, \cdots, a_n)\) we have the following inequalities:

\[
\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i a_i \geq x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \geq a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n.
\]

The next proof was summarized from part of a proof in [2] at p.66.

Proof. Without loss of generality, we may assume that \(a \leq b \leq c\). Therefore, \(a^2 \leq b^2 \leq c^2\), and \(\frac{1}{b} \geq \frac{1}{a} \geq \frac{1}{c}\). Applying rearrangement inequality,

\[
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} = a + b + c.
\]

Since it sometimes was mentioned in Vector Analysis or Linear Algebra courses, the next inequality may seem familiar for some students.

**Theorem 2.3** (Cauchy-Schwarz Inequality). Let \(a_1, a_2, \cdots, a_n\) and \(b_1, b_2, \cdots, b_n\) be real numbers. Then we have

\[
\left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right) \geq \left( \sum_{i=1}^{n} a_i b_i \right)^2.
\]

Proof. Applying Cauchy-Schwarz Inequality, we have

\[
(b + c + a) \left( \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \right) \geq (a + b + c)^2,
\]

which can be easily simplified to the claimed inequality by dividing \((a + b + c)\) at both sides.

The next inequality is very useful when considering sums of fractions.

**Theorem 2.4** (Radon’s Inequality). Let \(a_1, a_2, \cdots, a_n\) and \(b_1, b_2, \cdots, b_n\) be positive real numbers. If \(p\) is also a positive real number, then

\[
\frac{a_1^{p+1}}{b_1^p} + \frac{a_2^{p+1}}{b_2^p} + \cdots + \frac{a_n^{p+1}}{b_n^p} \geq \frac{(a_1 + a_2 + \cdots + a_n)^{p+1}}{(b_1 + b_2 + \cdots + b_n)^p}.
\]

In our next proof, we need to use a special case of Radon’s Inequality when \(p = 1\). This special case is sometimes referred to as Bergström’s Inequality (see [1]).

Proof. Applying Radon’s Inequality,

\[
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a + b + c)^2}{b + c + a} = a + b + c.
\]
The next proof requires an important but not so commonly used inequality.

**Definition 2.5.** We say that the sequence \((b_i)_{i=1}^n\) is majorized by \((a_i)_{i=1}^n\), denoted \((b_i) \prec (a_i)\), if we can rearrange the terms of the sequences \((a_i)\) and \((b_i)\) in such a way as to satisfy the following conditions:

1. \(b_1 + b_2 + \cdots + b_n = a_1 + a_2 + \cdots + a_n\).
2. \(b_1 \geq b_2 \geq \cdots \geq b_n\) and \(a_1 \geq a_2 \geq \cdots \geq a_n\).
3. \(b_1 + b_2 + \cdots + b_s \leq a_1 + a_2 + \cdots + a_s\) for any \(1 \leq s \leq n\).

**Theorem 2.6** (Karamata’s Inequality). Let \(f : I \to \mathbb{R}\) be a convex function on the interval \(I \subseteq \mathbb{R}\) and let \((a_i)_{i=1}^n, (b_i)_{i=1}^n\), where \(a_i, b_i \in I, i = 1, 2, \cdots, n\), are two sequences, such that \((a_i) \succ (b_i)\). Then

\[
f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).
\]

**Proof.** Let \(x_1 = \ln a, x_2 = \ln b,\) and \(x_3 = \ln c\). Then the claimed inequality becomes

\[
e^{2x_1-x_2} + e^{2x_2-x_3} + e^{2x_3-x_1} \geq e^{x_1} + e^{x_2} + e^{x_3}.
\]

Since \(f(x) = e^x\) is a convex function on \(R\), if we consider the sequences \((a_i) = \{2x_1, 2x_2, 2x_3, 2x_3 - x_1\}\) and \((b_i) = \{x_1, x_2, x_3\}\), we only need to prove that \((a_i) \succ (b_i)\) (ordered in some way) according to Karamata’s Inequality. Let us assume that \(2x_{m_1} - x_{m_1+1} \geq 2x_{m_2} - x_{m_2+1} \geq 2x_{m_3} - x_{m_3+1}\), and \(x_{k_1} \geq x_{k_2} \geq x_{k_3}\) for some indexes \(m, k, \in \{1, 2, 3\}\). Therefore,

\[
(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2+1}) \geq (2x_{k_1} - x_{k_1+1}) + (2x_{k_2} - x_{k_2+1}) \geq x_{k_1} + x_{k_2},
\]

and

\[
(2x_{m_1} - x_{m_1+1}) + (2x_{m_2} - x_{m_2+1}) + (2x_{m_3} - x_{m_3+1}) = x_1 + x_2 + x_3 = x_{k_1} + x_{k_2} + x_{k_3},
\]

So, \((a_i) \succ (b_i)\), and it finishes the proof.

**References**

Domination Number and Total Domination Number of Square of Normal Product of Cycles

Research Article

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Abstract: In 1958-Claude Berge introduced the domination number of a graph which is utilized to secure the single vertices. A set $S \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) - S$ is adjacent to at least one vertex of $S$. The cardinality of the smallest dominating set of $G$ is called the domination number of $G$. A dominating set $S$ is called total dominating set if the induced subgraph $\langle S \rangle$ has no isolated vertex. The square $G^2$ of a graph $G$ is obtained from $G$ by adding new edges between every two vertices having distance 2 in $G$. In this paper, we determine the domination number and total domination number of square of normal product of cycle graphs by evaluating their minimum dominating set and minimum total dominating set and short display are also provided to understand the results.

MSC: 05C50, 05C69

Keywords: Domination Number, Total Domination Number, Cycles, Square of a Graph, Normal Product.

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1. Introduction

Graph theory is a stand out applicable limb of advanced mathematics with its multi-pronged technological advancements. The beginning of extensive study of dominating sets took place in nearly 1960. De Jaenisch discovered and studied in depth the difficulties related to $(n \times n)$ chessboard problem that how many number of queens are required to lead. This seems the historical roots dating back to 1862 of this subject. Berge elaborated the conception of the domination number of a graph which is known as coefficient External Stability, in 1962. For the same concept, Ore provided the name “Dominating set” and “Domination number”. After this, many theories were revealed on graph theory. An interesting theory was revealed in 1977 by Cockayne and Hedetniemi, they made outstanding survey related to dominating set and total dominating set in graphs. The writing regarding this matter has been surveyed and point by point in the two excellent domination books by Hynes, Hedetniemi and Slater who made a remarkable showing with regards to of bringing together outcomes scattered through somewhere in the range of 1200 domination papers around then. They have applied the notation $\gamma_t(G)$ for the total domination number of a graph.

Domination applies in facility location problem like: Hospitals, Fire stations when a person needs to travel to get to the nearest facility, the number of amenities are permanent and its required to minimize the distance in one attempt. It also works on that problem where maximum space to a facility is permanent and needed one attempt to minimize the number of amenities compulsory. In case that everyone is serviced Concepts from domination is also occur in problems including

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discovering sets of representative in the administering communication or electrical network and in land reconnoitering i.e. an evaluator must remain in order to acquire the estimations of height for a whole domain meanwhile the number of places are minimized. This is a fact that mathematical study of domination is creating its own area of study and plays an important role in mathematical study. It is surely possible that undiscovered prospect of possible set of the combinations of graphs may become a key aspect of network design. A productive endeavour could be significant in the study of different methods of creating a network, different products of graphs as an example.

All graphs consider to be simple, finite and undirected throughout the paper. Graph $G = (V, E)$ [13] where $V = V(G)$ is the vertex set and $E = E(G)$ edge set. Given two vertices $u$ and $v$ in graph $G$, we say $u$ dominates $v$ if $u = v$ or $uv \in E$. In graph theory, many graphs has been awared but we will discuss only Cycle graph with $n$ vertices i.e. $C_n$ and Path graph with $n$ vertices i.e. $P_n$ [10]. “The open neighborhood of a vertex $u$ is denoted by $N(u) = \{v \in V(G), uv \in E(G)\}$ and the close neighborhood of a vertex $u$ is denoted by $N[u] = N(u) \cup \{u\}$. The open neighborhood and closed neighborhood of $S \subseteq V(G)$ are defined as $N(S) = \cup_{u \in S} N(u)$ and $N[S] = \cup_{u \in S} N[u]$”. A set $S \subseteq V(G)$ is a dominating set of graph $G$ if every vertex of $V(G) - S$ is adjacent to atleast one vertex of set $S$. Domination number is called the cardinality of the smallest dominating set of $G$ which is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma-$ set. “A total dominating set, denoted $TDS$, of $G$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$ is the minimum cardinality of a $TDS”$. Clearly $\gamma(G) \leq \gamma_t(G)$, also it has been proved that $\gamma_t(G) \leq 2 \gamma(G)$. For a survey of total domination in graphs, See [8].

Given any graph $G$, its square graph $G^2$ is a graph with the same vertex set $V(G)$ and two vertices are adjacent whenever they are at distance 1 or 2 in graph $G$. A set $S \subseteq V(G)$ is a 2− distance dominating set of $G$ if $d_G(u, s) = 1$ or 2 for each vertex of $\{V(G) - S\}$. $\gamma^2(G)$ is denoted as the cardinality of the smallest two distance dominating set of $G$ and is called 2− distance domination number of $G$. In graph theory, many operations will be used like as: Normal product, Cartesian product, Tensor product, Corona product, etc. but here bring to light on the “Normal Product”. The Normal product was first introduced by Sabidussi [11]. The normal product of a graphs $G(V(G), E(G))$ and $H(V(H), E(H))$ denoted by $G \boxtimes H$ is a graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$, that is, the set $\{(u, v) | u \in G, v \in H\}$ and edge $((u, u'), (v, v'))$ exists whenever any of the following conditions hold:

1. $(u, u') \in E(G)$ and $v = v'$
2. $u = u'$ and $(v, v') \in E(H)$
3. $(u, u') \in E(G)$ and $(v, v') \in E(H)$.

In this paper, we provide an approach to finding Dominating set and Total Dominating Set of Square of Normal Product of Cycle Graph. It is based on the cardinality of the smallest dominating set and smallest total dominating set of $G$. This paper is organized as follows: In Section 2, related work is given. Section 3 and Section 4 contains the main result of the paper and its standard graphs shown by example. In the Section 5, discussion and conclusion is given. Section 6 contains references.

2. Related Work

Haynes et al. [6] examined the dominating number of distinct graphs and indicated various problems in this concept. Further, Jacobson and Kinch [9] proposed the domination number of products of graphs. It is based on the different products of graphs and took different approaches to the problem. Chaluvaraju and Appajigowda [3] established the dominating set of normal product of paths and cycles. They identified the various results of normal product of paths and its dominating sets.
Alishahi et al. [1] deduced the square of graphs mainly and contradicted an important resolution regarding the Cartesian product of cycles and paths. Atapour M. et al. [2] gave important results over total dominating sets of different graphs which was used in their paper. Some results at high level and algorithms for total dominating set were analysed by Henning [8]

3. Domination Number of \((C_m \boxtimes C_n)^2\)

In this section, We investigate the domination number of square of normal products including two cycles \(C_m\) and \(C_n\) for \(m \geq 3\) and \(n \geq 3\), respectively. It is easy to check the following:

**Theorem 3.1.** For \(m = 5k_1 - 2\) or \(5k_1 - 1\) or \(5k_1\), \(k_1 \geq 1\) and \(m = 5k_1 - 3\), \(k_1 \geq 2\), \(\gamma[(C_m \boxtimes C_n)]^2 = k_1k_2\), if \(n = 5k_2 - 4\) or \(5k_2 - 3\) or \(5k_2 - 2\) or \(5k_2 - 1\) or \(5k_2\), \(k_2 \geq 1\).

**Proof.** Let \(C_m\) and \(C_n\) be the Cycle graphs with vertex sets \(\{u_1, u_2, ..., u_m\}\) and \(\{v_1, v_2, ..., v_n\}\) respectively. Then the following cases arise.

**Case 1:** For \(m = 5k_1 - 2\) or \(5k_1 - 1\) or \(5k_1\), \(k_1 \geq 1\). In this case, we consider the two subcases as follows:

**Subcase 1:** For \(n = 5k_2 - 2\) or \(5k_2 - 1\) or \(5k_2\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset \(A = \{(u_{5t_1-2}, v_{5t_2-2}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)- set. Clearly, no dominating set with cardinality less than \(|A|\) exists in the graph. Thus \(A\) is minimum dominating set of \([C_m \boxtimes C_n]^2\). Hence, \(\gamma[(C_m \boxtimes C_n)]^2 = |A| = k_1k_2\).

**Subcase 2:** For \(n = 5k_2 - 4\) or \(5k_2 - 3\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset \(B = \{(u_{5t_1-3}, v_{5t_2-4}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)- set. Clearly, no dominating set with cardinality less than \(|B|\) exists in the graph. Thus \(B\) is minimum dominating set of \([C_m \boxtimes C_n]^2\). Hence, \(\gamma[(C_m \boxtimes C_n)]^2 = |B| = k_1k_2\).

**Case 2:** For \(m = 5k_1 - 3\), \(k_1 \geq 2\). In this case, we consider the two subcases as follows:

**Subcase 1:** For \(n = 5k_2 - 2\) or \(5k_2 - 1\) or \(5k_2\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset \(C = \{(u_{5t_1-3}, v_{5t_2-2}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)- set. Clearly, no dominating set with cardinality less than \(|C|\) exists in the graph. Thus \(C\) is minimum dominating set of \([C_m \boxtimes C_n]^2\). Hence, \(\gamma[(C_m \boxtimes C_n)]^2 = |C| = k_1k_2\).

**Subcase 2:** For \(n = 5k_2 - 4\) or \(5k_2 - 3\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset \(D = \{(u_{5t_1-3}, v_{5t_2-4}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)- set. Clearly, no dominating set with cardinality less than \(|D|\) exists in the graph. Thus \(D\) is minimum dominating set of \([C_m \boxtimes C_n]^2\). Hence, \(\gamma[(C_m \boxtimes C_n)]^2 = |D| = k_1k_2\).
Figure 2: A dominating set for \((C_m \boxtimes C_n)^2\) (Case 2)

**Theorem 3.2.** For \(m = 5k_1 - 4\), \(k_1 \geq 2\),

\[
\gamma[(C_m \boxtimes C_n)^2] = \begin{cases} 
    k_1k_2 & \text{if } n = 5k_2 - 3 \text{ or } 5k_2 - 2 \text{ or } 5k_2 - 1 \text{ or } 5k_2, \ k_2 \geq 1 \\
    k_1(k_2 + 1) - 1 & \text{if } n = 5k_2 + 1, \ k_2 \geq 1.
\end{cases}
\]

**Proof.** Let \(C_m\) and \(C_n\) be the Cycle graphs with vertex sets \(\{u_1, u_2, \ldots, u_m\}\) and \(\{v_1, v_2, \ldots, v_n\}\) respectively. Then the following cases arise.

**Case 1:** If \(m = 5k_1 - 4\), \(k_1 \geq 2\). In this case, we consider the two subcases as follows:

**Subcase 1:** For \(n = 5k_2 - 3\) or \(5k_2 - 2\), \(k_2 \geq 1\). In \([(C_m \boxtimes C_n)^2]^{\gamma}\), the subset \(A = \{(u_{3t_1}, v_{5t_2-4}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)-set. Clearly, no dominating set with cardinality less than \(|A|\) exists in the graph. Thus \(A\) is minimum dominating set of \([(C_m \boxtimes C_n)^2]^{\gamma}\). Hence, \(\gamma[(C_m \boxtimes C_n)^2] = |A| = k_1k_2\).

**Subcase 2:** For \(n = 5k_2 - 1\) or \(5k_2\), \(k_2 \geq 1\). In \([(C_m \boxtimes C_n)^2]^{\gamma}\), the subset \(B = \{(u_{3t_1}, v_{5t_2-2}) : 1 \leq t_1 \leq k_1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)-set. Clearly, no dominating set with cardinality less than \(|B|\) exists in the graph. Thus \(B\) is minimum dominating set of \([(C_m \boxtimes C_n)^2]^{\gamma}\). Hence, \(\gamma[(C_m \boxtimes C_n)^2] = |B| = k_1k_2\).

**Case 2:** If \(m = 5k_1 - 4\), \(k_1 \geq 2\). In this case, we consider the two subcases as follows:

**Subcase 1:** For \(n = 6\), \(k_2 \geq 2\). In \([(C_m \boxtimes C_n)^2]^{\gamma}\), the subset \(C = \{(u_1, v_1)\} \cup \{(u_{5t_1+1}, v_3) : 1 \leq t_1 \leq k_1 - 1\} \cup \{(u_{5t_2+2}, v_{5t_2+1}) : 1 \leq t_1 \leq k_1 - 1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)-set. Clearly, no dominating set with cardinality less than \(|C|\) exists in the graph. Thus \(C\) is minimum dominating set of \([(C_m \boxtimes C_n)^2]^{\gamma}\). Hence, \(\gamma[(C_m \boxtimes C_n)^2] = |C| = (k_1(k_2 + 1) - 1)\).

**Subcase 2:** For \(n = 5k_2 + 1\), \(k_2 \geq 2\). In \([(C_m \boxtimes C_n)^2]^{\gamma}\), the subset \(D = \{(u_1, v_1)\} \cup \{(u_{5t_1+1}, v_3) : 1 \leq t_1 \leq k_1 - 1\} \cup \{(u_{5t_1-2}, v_{5t_2+1}) : 1 \leq t_1 \leq k_1 - 1; 1 \leq t_2 \leq k_2\} \cup \{(u_{5t_1+1}, v_{5t_2+3}) : t_1 = k_1 - 1; 1 \leq t_2 \leq k_2\}\) is the \(\gamma\)-set. Clearly, no dominating set with cardinality less than \(|D|\) exists in the graph. Thus \(D\) is minimum dominating set of \([(C_m \boxtimes C_n)^2]^{\gamma}\). Hence, \(\gamma[(C_m \boxtimes C_n)^2] = |D| = (k_1(k_2 + 1) - 1)\).
For \( i \leq k \leq 2 \) or \( 7k_2 - 2 \) or \( 7k_2 - 1 \) or \( 7k_2 \), \( k \geq 1 \).

In [\( (C_m \boxtimes C_n) \)], the subset \( A = \{ (u_3, v_{7t-6}) : 1 \leq t \leq k_2 \} \cup \{ (u_3, v_{7t-4}) : 1 \leq t \leq k_2 \} \) is the \( \gamma_{i} \) set. Clearly, no total dominating set with cardinality less than \( |A| \) exists in the graph. Thus \( A \) is minimum total dominating set of [\( (C_m \boxtimes C_n) \)]. Hence, \( \gamma_{i}[(C_m \boxtimes C_n)]^2 = |A| = 2k_1k_2 \).

Subcase 2: For \( n = 7k_2 - 1 \) or \( 7k_2, k_2 \geq 1 \). In [\( (C_m \boxtimes C_n) \)], the subset \( B = \{ (u_3, v_{7t-6}) : 1 \leq t \leq k_2 \} \cup \{ (u_3, v_{8t-5}) : 1 \leq t \leq k_2 \} \) is the \( \gamma_{i} \) set. Clearly, no total dominating set with cardinality less than \( |B| \) exists in the graph. Thus \( B \) is minimum total dominating set of [\( (C_m \boxtimes C_n) \)]. Hence, \( \gamma_{i}[(C_m \boxtimes C_n)]^2 = |B| = 2k_1k_2 \).

Case 2: If \( m = 5k_1 - 2 \) or \( 5k_1 - 1 \) or \( 5k_1 \), \( k_1 = 1 \). In this case, For \( n = 7k_2 + 1 \) or \( 7k_2 + 2, k_2 \geq 1 \). In [\( (C_m \boxtimes C_n) \)], the subset \( C = \{ (u_3, v_{8t-6}) : 1 \leq t \leq k_2 \} \cup \{ (u_3, v_{8t-5}) : 1 \leq t \leq k_2 \} \) is the \( \gamma_{i} \) set. Clearly, no total dominating set with cardinality less than \( |C| \) exists in the graph. Thus \( C \) is minimum total dominating set of [\( (C_m \boxtimes C_n) \)]. Hence, \( \gamma_{i}[(C_m \boxtimes C_n)]^2 = |C| = 2k_1k_2 + 1 \).
Theorem 4.2. For \( m = 5k_1 + 1 \) or \( 5k_1 + 2 \), \( k_1 = 1 \),

\[
\gamma_t[(C_m \Box C_n)^2] = \begin{cases} 
2 & \text{if } n = 3 \text{ or } 4 \text{ or } 5 \\
2k_2 & \text{if } n = 5k_2 + 1 \text{ or } 5k_2 + 2 \text{ or } 5k_2 + 3 \text{ or } 5k_2 + 4 \text{ or } 5k_2 + 5, \ k_2 \geq 1.
\end{cases}
\]

Proof. Let \( C_m \) and \( C_n \) be the Cycle graphs with vertex sets \( \{u_1, u_2, ..., u_m\} \) and \( \{v_1, v_2, ..., v_n\} \) respectively. Then the following cases arise.

Case 1: For \( n = 3 \) or \( 4 \) or \( 5 \). In \( [(C_m \Box C_n)^2] \), the subset \( A = \{(u_3, v_1), (u_3, v_3)\} \) is the \( \gamma_t \) set. Clearly, no total dominating set with cardinality less than \( |A| \) exists in the graph. Thus \( A \) is minimum total dominating set of \( [(C_m \Box C_n)^2] \). Hence, \( \gamma_t[(C_m \Box C_n)^2] = |A| = 2 \).

Case 2: If \( m = 5k_1 + 1 \) or \( 5k_1 + 2 \), \( k_1 = 1 \). In this case, we consider the two subcases as follows:

Subcase 1: For \( n = 5k_2 + 1 \) or \( 5k_2 + 2 \), \( k_2 \geq 1 \). In \( [(C_m \Box C_n)^2] \), the subset \( B = \{(u_5, v_{5t-4}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{5t-4}) ; 1 \leq t \leq k_2 + 1\} \) is the \( \gamma_t \) set. Clearly, no total dominating set with cardinality less than \( |B| \) exists in the graph. Thus \( B \) is minimum total dominating set of \( [(C_m \Box C_n)^2] \). Hence, \( \gamma_t[(C_m \Box C_n)^2] = |B| = 2k_2 \).

Subcase 2: For \( n = 5k_2 + 3 \) or \( 5k_2 + 4 \) or \( 5k_2 + 5 \), \( k_2 \geq 1 \). In \( [(C_m \Box C_n)^2] \), the subset \( C = \{(u_5, v_{5t-2}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{5t-2}) ; 1 \leq t \leq k_2 + 1\} \) is the \( \gamma_t \) set. Clearly, no total dominating set with cardinality less than \( |C| \) exists in the graph. Thus \( C \) is minimum total dominating set of \( [(C_m \Box C_n)^2] \). Hence, \( \gamma_t[(C_m \Box C_n)^2] = |C| = 2k_2 \).

Theorem 4.3. For \( m = 5k_1 + 3 \) or \( 5k_1 + 4 \), \( k_1 = 1 \),

\[
\gamma_t[(C_m \Box C_n)^2] = \begin{cases} 
3k_1k_2 & \text{if } n = 5k_2 - 2 \text{ or } 5k_2 - 1 \text{ or } 5k_2, \ k_2 = 1 \text{ or } 2 \\
4 & \text{if } n = 6 \text{ or } 7 \\
k_2(k_1 + 2) + 3 & \text{if } n = 5k_2 + 1 \text{ or } 5k_2 + 2 \text{ or } 5k_2 + 3 \text{ or } 5k_2 + 4 \text{ or } 5k_2 + 5, \ k_2 \geq 2.
\end{cases}
\]
Proof. Let $C_m$ and $C_n$ be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: For $n = 5k_2 - 2$ or $5k_2 - 1$ or $5k_2$, $k_2 = 1$ or 2. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_{5t-2}) ; 1 \leq t \leq k_2\} \cup \{(u_5, v_{5t-2}) ; 1 \leq t \leq k_2\} \cup \{(u_7, v_{5t-2}) ; 1 \leq t \leq k_2\}$ is the $\gamma_t$-set. Clearly, no total dominating set with cardinality less than $|A|$ exists in the graph. Thus $A$ is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |A| = 3k_1k_2$.

Case 2: For $n = 6$ or 7. In $[(C_m \boxtimes C_n)]^2$, the subset $B = \{(u_3, v_{2t+1}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{2t+1}) ; 1 \leq t \leq k_2 + 1\}$ is the $\gamma_t$-set. Clearly, no total dominating set with cardinality less than $|B|$ exists in the graph. Thus $B$ is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |B| = 4$.

Case 3: If $m = 5k_1 + 3$ or $5k_1 + 4$, $k_1 = 1$. In this case, we consider the two subcases as follows:

Subcase 1: For $n = 5k_2 + 1$ or $5k_2 + 2$ or $5k_2 + 3$, $k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $C = \{(u_3, v_{5t-6}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{5t-6}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_7, v_{5t-6}) ; 1 \leq t \leq k_2 + 1\}$ is the $\gamma_t$-set. Clearly, no total dominating set with cardinality less than $|C|$ exists in the graph. Thus $C$ is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |C| = k_2(k_1 + 2) + 3$.

Subcase 2: For $n = 5k_2 + 4$ or $5k_2 + 5$, $k_2 \geq 2$. In $[(C_m \boxtimes C_n)]^2$, the subset $D = \{(u_3, v_{5t-2}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{5t-2}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_7, v_{5t-2}) ; 1 \leq t \leq k_2 + 1\}$ is the $\gamma_t$-set. Clearly, no total dominating set with cardinality less than $|D|$ exists in the graph. Thus $D$ is minimum total dominating set of $[(C_m \boxtimes C_n)]^2$. Hence, $\gamma_t[(C_m \boxtimes C_n)]^2 = |D| = k_2(k_1 + 2) + 3$.

\[\gamma_t[(C_m \boxtimes C_n)]^2 = \begin{cases} (k_1+1)(k_2+1) & \text{if } n = 2k_2 + 4 \text{ or } 2k_2 + 5, \text{ } k_2 \geq 1. \end{cases}\]

\[\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}\]

Figure 6: A total dominating set for $(C_m \boxtimes C_n)^2$


Theorem 4.4. For $m = 5k_1 + 1$ or $5k_1 + 2$ or $5k_1 + 3$ or $5k_1 + 4$ or $5k_1 + 5$, $k_1 \geq 2$,

\[\gamma_t[(C_m \boxtimes C_n)]^2 = \begin{cases} (k_1+1)(k_2+1) & \text{if } n = 2k_2 + 4 \text{ or } 2k_2 + 5, \text{ } k_2 \geq 1. \end{cases}\]

Proof. Let $C_m$ and $C_n$ be the Cycle graphs with vertex sets $\{u_1, u_2, ..., u_m\}$ and $\{v_1, v_2, ..., v_n\}$ respectively. Then the following cases arise.

Case 1: If $m = 5k_1 + 1$ or $5k_1 + 2$ or $5k_1 + 3$ or $5k_1 + 4$ or $5k_1 + 5$, $k_1 \geq 2$. In this case, we consider the four subcases as follows:

Subcase 1: For $m = 5k_1 + 1$ or $5k_1 + 2$, $k_1 \geq 2$ and $n = 2k_2 + 4$, $\geq 1$. In $[(C_m \boxtimes C_n)]^2$, the subset $A = \{(u_3, v_{2t}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{2t-1}) ; 1 \leq t \leq k_2 + 1\} \cup \{(u_7, v_{2t-1}) ; 1 \leq t \leq k_2 + 1\}$ is the $\gamma_t$-set. Clearly, no total dominating
set with cardinality less than \(|A|\) exists in the graph. Thus \(A\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |A| = (k_1 + 1)(k_2 + 1).
\]

**Subcase 2:** For \(m = 5k_1 + 3\) or \(5k_1 + 4\) or \(5k_1 + 5\), \(k_1 \geq 2\) and \(n = 2k_2 + 4\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset 
\[
B = \{(u_3, v_{2t}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{2t}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t}) : 1 \leq t \leq k_2 + 1\}
\]
is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|B|\) exists in the graph. Thus \(B\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |B| = (k_1 + 1)(k_2 + 1).
\]

**Subcase 3:** For \(m = 5k_1 + 1\) or \(5k_1 + 2\), \(k_1 \geq 2\) and \(n = 2k_2 + 5\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset 
\[
C = \{(u_3, v_{2t+1}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{2t+1}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t+1}) : 1 \leq t \leq k_2 + 1\}
\]
is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|C|\) exists in the graph. Thus \(C\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |C| = (k_1 + 1)(k_2 + 1).
\]

**Subcase 4:** For \(m = 5k_1 + 3\) or \(5k_1 + 4\) or \(5k_1 + 5\), \(k_1 \geq 2\) and \(n = 2k_2 + 5\), \(k_2 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset 
\[
D = \{(u_3, v_{2t+1}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_5, v_{2t+1}) : 1 \leq t \leq k_2 + 1\} \cup \{(u_{11}, v_{2t+1}) : 1 \leq t \leq k_2 + 1\}
\]
is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|D|\) exists in the graph. Thus \(D\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |D| = (k_1 + 1)(k_2 + 1).
\]

**Theorem 4.5.** For \(n = 3\) or \(4\) or \(5\), 
\[
\gamma_t([C_m \boxtimes C_n]^2) = \begin{cases} 
2(k_1 + 1) & \text{if } m = 7k_1 + 3 \text{ or } 7k_1 + 4 \text{ or } 7k_1 + 5 \text{ or } 7k_1 + 6 \text{ or } 7k_1 + 7, \ k_1 \geq 1 \\
2k_1 + 3 & \text{if } m = 7k_1 + 8 \text{ or } 7k_1 + 9, \ k_1 \geq 1.
\end{cases}
\]

**Proof.** Let \(C_m\) and \(C_n\) be the Cycle graphs with vertex sets \(\{u_1, u_2, \ldots, u_m\}\) and \(\{v_1, v_2, \ldots, v_n\}\) respectively. Then the following cases arise.

**Case 1:** If \(n = 3\) or \(4\) or \(5\). In this case, we consider the two subcases as follows:

**Subcase 1:** For \(m = 7k_1 + 3\) or \(7k_1 + 4\), \(k_1 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset 
\[
A = \{(u_3, v_5) \cup (u_5, v_3) \cup (u_{7t+1}, v_3) : 1 \leq t \leq k_1\} \cup \{(u_{7t+3}, v_3) : 1 \leq t \leq k_1\}
\]
is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|A|\) exists in the graph. Thus \(A\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |A| = 2(k_1 + 1).
\]

**Subcase 2:** For \(m = 7k_1 + 5\) or \(7k_1 + 6\) or \(7k_1 + 7, k_1 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset 
\[
B = \{(u_3, v_3) \cup (u_5, v_3) \cup (u_{7t+1}, v_3) : 1 \leq t \leq k_1\} \cup \{(u_{7t+4}, v_3) : 1 \leq t \leq k_1\}
\]
is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|B|\) exists in the graph. Thus \(B\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |B| = 2k_1 + 3.
\]

**Case 2:** If \(n = 3\) or \(4\) or \(5\), For \(m = 7k_1 + 8\) or \(7k_1 + 9\), \(k_1 \geq 1\). In \([C_m \boxtimes C_n]^2\), the subset \(C = \{(u_3, v_5) \cup (u_5, v_3) \cup (u_{7t+1}, v_3) : 1 \leq t \leq k_1\} \cup \{(u_{7t+3}, v_3) : 1 \leq t \leq k_1\}\) is the \(\gamma_t\)-set. Clearly, no total dominating set with cardinality less than \(|C|\) exists in the graph. Thus \(C\) is minimum total dominating set of \([C_m \boxtimes C_n]^2\). Hence, 
\[
\gamma_t([C_m \boxtimes C_n]^2) = |C| = 2(k_1 + 1).
\]

Figure 7: A total dominating set for \((C_m \boxtimes C_n)^2\)
\{(u_3, v_3) \cup (u_5, v_3) \cup (u_7, v_3) ; 1 \leq t \leq k_1 \} \cup \{(u_7t+7, v_3) ; 1 \leq t \leq k_1 \} is the \( \gamma_t \) set. Clearly, no total dominating set with cardinality less than \(|C|\) exists in the graph. Thus \( C \) is minimum total dominating set of \([(C_m \boxtimes C_n)^2]\). Hence, \( \gamma_t[(C_m \boxtimes C_n)^2] = |C| = 2k_1 + 3 \).

Figure 8: A total dominating set for \((C_m \boxtimes C_n)^2\)

5. Conclusions

In this paper, an approach is presented to find the domination number and the total domination number of square of normal product of cycles. Our main emphasis is an obtaining the domination number and the total domination number which are related to square of normal product of cycle graphs, that give the main results of \((C_m \boxtimes C_n)^2\).

References


Hyers Type Stability of a Radical Reciprocal Quadratic Functional Equation Originating From 3 Dimensional Pythagorean Means

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Abstract: In this paper, authors introduce a 3 dimensional Pythagorean mean functional equation

\[ f \left( \sqrt{x^2 + y^2} \right) + f \left( \sqrt{y^2 + z^2} \right) + f \left( \sqrt{z^2 + x^2} \right) = \frac{f(x)f(y)}{f(x)+f(y)} + \frac{f(y)f(z)}{f(y)+f(z)} + \frac{f(z)f(x)}{f(z)+f(x)} \]

which relates the three classical Pythagorean mean and investigate its generalized Hyers-Ulam stability.

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Keywords: Pythagorean Means, Arithmetic mean, Geometric mean and Harmonic mean, Generalized Hyers-Ulam stability.

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1. Introduction

The stability problem of functional equations originates from the basic question of S.M. Ulam [26] in 1940. The initial clarification to Ulam’s query was given by D.H. Hyers [13]. He considered the case of approximately additive mappings. Further between 1951 to 2007, T. Aoki [2], Th.M. Rassias [23], J.M.Rassias [21], P.Gavruta [10] and J.M.Rassias et.al., [25] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded. For the past seven decades, the generalized Ulam stability for various functional equations have been extensively investigated by numerous authors; one can refer to [3–5, 7, 9, 12, 14–19, 22, 24]. There is a legend that one day when Pythagoras (c.500 BCE) was passing a blacksmiths shop, he heard harmonious music ringing from the hammers. When he enquired, he was told that the weights of the hammers were 6, 8, 9, and 12 pounds. These ratios produce a fundamental and its fourth, fifth and octave. This was evidence that the elegance of mathematics is manifested in the harmony of nature. Returning to music, these ratios are indeed a foundation of music as noted by Archytus of Tarentum (c.350 BCE): There are three means in music: one is the arithmetic, the second is the geometric and the third is the sub contrary, which they call harmonic. The arithmetic mean is when there are three terms showing successively the same excess: the second exceeds the third by the same amount as the first exceeds the second. In this proportion, the ratio of the larger number is less, that of the smaller

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numbers greater. The geometric mean is when the second is to the third as the first is to the second; in this, the greater numbers have the same ratio as the smaller numbers. The sub contrary, which we call harmonic, is as follows: by whatever part of itself the first term exceeds the second, the middle term exceeds the third by the same part of the third. In this proportion, the ratio of the larger numbers is larger and of the lower numbers less [8].

**Definition 1.1 (Pythagorean Means [6]).** In Mathematics, the three classical Pythagorean means are the Arithmetic Mean (A.M.), the Geometric Mean (G.M.), and the Harmonic Mean (H.M.). They are defined for $N$ values by

1. **Arithmetic Mean (A.M.):** Arithmetic mean is the total of all the items divided by their total number of items.

   $$A.M. = \frac{X_1 + X_2 + \cdots + X_N}{N}.$$  

2. **Geometric Mean (G.M.):** Geometric mean of $N$ values is the $N$th root of the product of $N$ values.

   $$G.M. = \sqrt[N]{X_1 \cdot X_2 \cdots X_N}.$$

3. **Harmonic Mean (H.M.):** Harmonic mean is the reciprocal of the means of the reciprocals of the values.

   $$H.M. = \frac{N}{\frac{1}{X_1} + \frac{1}{X_2} + \cdots + \frac{1}{X_N}}.$$

**Lemma 1.2.** For any two items $a$ and $b$, we have $G.M. = \sqrt{A.M. \times H.M.}$.

**Proof.** Let $a$ and $b$ be two items. Then, we have

\begin{align*}
A.M. &= \frac{a + b}{2}, \quad (1) \\
G.M. &= \sqrt{ab}, \quad (2) \\
H.M. &= \frac{2}{\frac{1}{a} + \frac{1}{b}}. \quad (3)
\end{align*}

From (1) and (3), we arrive

\begin{align*}
A.M. \times H.M. &= \frac{a + b}{2} \times \frac{2}{\frac{1}{a} + \frac{1}{b}} \\
&= \frac{a + b}{2} \times \frac{2}{\frac{a + b}{ab}} \\
&= \frac{a + b}{2} \times \frac{2ab}{a + b} \\
&= ab \\
&= (G.M.)^2.
\end{align*}

Hence we derived our result. \hfill \Box

### 2. Geometrical Interpretation of Functional Equation

Let $O$ be the center and $X, Y, Z$ be any point on the three perpendicular axes. Assume that $OX = b, OY = a, OZ = c$. 
From the Triangle $XOY$, $YOZ$ and $XOZ$, we have by Pythagoras Theorem

$$YX^2 = OY^2 + OX^2 = a^2 + b^2 \Rightarrow YX = \sqrt{a^2 + b^2}, \quad (4)$$

$$ZY^2 = OY^2 + OZ^2 = a^2 + c^2 \Rightarrow ZY = \sqrt{a^2 + c^2}, \quad (5)$$

$$XZ^2 = OZ^2 + OX^2 = c^2 + b^2 \Rightarrow XZ = \sqrt{c^2 + b^2}. \quad (6)$$

Adding (4), (5) and (6), we obtain

$$YX + ZY + XZ = \sqrt{a^2 + b^2} + \sqrt{a^2 + c^2} + \sqrt{c^2 + b^2}. \quad (7)$$

The above equation can be transformed into a radical reciprocal quadratic functional equation of the following form

$$f(\sqrt{x^2 + y^2}) + f(\sqrt{y^2 + z^2}) + f(\sqrt{z^2 + x^2}) = \frac{f(x)f(y)}{f(x) + f(y)} + \frac{f(y)f(z)}{f(y) + f(z)} + \frac{f(z)f(x)}{f(z) + f(x)} \quad (8)$$

having solution

$$f(x) = \frac{k}{x^2} \quad (9)$$

for any constant $k$.

### 3. General Solution of the Functional Equation (8)

In this section, motivated by the work of Roman Ger [11], we present the general solution of the Pythagorean mean functional equation in the simplest case and also we give the differentiable solution of (8). The following Theorem gives the solution of (8) in the simplest case.

**Theorem 3.1.** The only nonzero solution of a function $f : (0, \infty) \to \mathbb{R}$, admitting a finite limit of the quotient $\frac{f(x)}{x}$ at zero, of the equation (8) is of the form $\frac{k}{x^2}$ for all $x, y \in (0, \infty)$.

**Proof.** Put $x = y = z = x$ (8), we obtain

$$3f\left(\sqrt{2x}\right) = 3 \cdot \frac{1}{2f(x)} \Rightarrow f\left(\sqrt{2x}\right) = \frac{1}{2f(x)} \quad (10)$$

for all $x \in (0, \infty)$. The rest of the proof is similar to that of [20].

The following Theorem gives the differentiable solution of the Pythagorean mean functional equation (8).

**Theorem 3.2.** Let $f : (0, \infty) \to \mathbb{R}$ be continuously differentiable functions with nowhere vanishing derivatives $f'$. Then $f$ yields a solution to the functional equation (8) if and only if there exists nonzero real constants $k$ such that $\frac{k}{x^2}$ for all $x, y \in (0, \infty)$. 

\[\square\]
Proof. Differentiate equation (8) with respect to $x$ on both sides, we obtain
\begin{align*}
f'( \sqrt{x^2 + y^2} ) \frac{x}{\sqrt{x^2 + y^2}} + f'( \sqrt{z^2 + x^2} ) \frac{x}{\sqrt{z^2 + x^2}} &= \frac{f'(x)f'(y)}{(f(x) + f(y))^2} + \frac{f'(z)f'(x)}{(f(z) + f(x))^2} \\
(11)
\end{align*}

for all $x, y, z \in (0, \infty)$. Setting $x = y = z = x$ in (8) and (11), we get
\begin{align*}
f(\sqrt{2}x) &= \frac{1}{2}f(x) \\
f'(\sqrt{2}x) &= \frac{1}{2\sqrt{2}}f'(x) \\
(12) \hspace{1cm} (13)
\end{align*}

for all $x \in (0, \infty)$. Replacing $y = z = \sqrt{2}x$ in (11) and using (12), (14), we arrive
\begin{align*}
f'(\sqrt{3}x) &= \frac{1}{3\sqrt{3}}f'(x) \\
(14)
\end{align*}

for all $x \in (0, \infty)$. The rest of the proof is similar to that of [20]. \hfill \Box


Through out section, let $E$ be a linear space and $F$ be a Banach space. Define a difference operation $Df(x, y, z)$ by
\begin{align*}
Df(x, y, z) = f\left( \sqrt{x^2 + y^2} \right) + f\left( \sqrt{y^2 + z^2} \right) + f\left( \sqrt{z^2 + x^2} \right) - f(x)f(y) - f(y)f(z) - f(z)f(x) \\
(15)
\end{align*}

for all $x, y, z \in E$. Now, we investigate the generalized Hyers-Ulam stability of the functional equation (8) in Banach space using direct method.

**Theorem 4.1.** If $f : E \to F$ be a function satisfying the functional inequality
\begin{align*}
\|Df(x, y, z)\| \leq A(x, y, z) \\
(15)
\end{align*}

where $A : E^3 \to [0, \infty)$ be a function such that
\begin{align*}
\lim_{m \to \infty} 2^{m_j}A \left( 2^{\frac{m_i}{4}} x, 2^{\frac{m_j}{4}} y, 2^{\frac{m_k}{4}} z \right) = 0 \\
(16)
\end{align*}

for all $x, y, z \in E$. Then there exists a unique radical reciprocal quadratic function $Q : E \to F$ satisfying the functional equation (8) and the inequality
\begin{align*}
\|f(x) - Q(x)\| \leq \frac{2}{3} \sum_{i=1}^{\infty} 2^{ij}A \left( 2^{\frac{ij}{4}} x, 2^{\frac{ij}{4}} y, 2^{\frac{ij}{4}} z \right) \\
(17)
\end{align*}

for all $x \in E$, where $j = \pm 1$. The function $Q(x)$ is defined as
\begin{align*}
Q(x) = \lim_{m \to \infty} 2^{m_j}f \left( 2^{\frac{m_i}{4}} x \right), \text{ for all } x \in E. \\
(18)
\end{align*}

**Proof.** Replacing $(x, y, z)$ by $(x, x, x)$ in (15), we get
\begin{align*}
\left\| 3f\left( \sqrt{2}x \right) - \frac{3f(x)}{2} \right\| \leq A(x, x, x) \\
(19)
\end{align*}
for all $x \in E$. It follows from (19), we have
\[
\left\| 2f \left( \sqrt{2} x \right) - f(x) \right\| \leq \frac{2}{3} A(x, x, x)
\] (20)
for all $x \in E$. Setting $x$ by $\sqrt{2} x$ and multiply by 2 in (20), we arrive
\[
\left\| 2^2 f \left( \sqrt{2} x \right) - 2f(\sqrt{2} x) \right\| \leq \frac{4}{3} A \left( \sqrt{2} x, \sqrt{2} x, \sqrt{2} x \right)
\] (21)
for all $x \in E$. With the use of triangle inequality it follows from (20) and (21), we obtain
\[
\left\| 2^2 f \left( \sqrt{2} x \right) - f(x) \right\| \leq \frac{2}{3} \left[ A(x, x, x) + 2A \left( \sqrt{2} x, \sqrt{2} x, \sqrt{2} x \right) \right]
\] (22)
for all $x \in E$. Generalizing, for any positive integer $m$, one can reach
\[
\left\| 2^m f \left( 2^\frac{m}{2} x \right) - f(x) \right\| \leq \frac{2}{3} \sum_{i=0}^{m-1} 2^i A \left( 2^k x, 2^k x, 2^k x \right)
\] (23)
for all $x \in E$. Thus the sequence $\{2^m f \left( 2^\frac{m}{2} x \right) \}$ is a Cauchy sequence $F$. Indeed, replacing $x$ by $2^\frac{m}{2} x$ and multiply by $2^n$ in (24) and using (16), we have
\[
\left\| 2^{m+n} f \left( 2^{\frac{m+n}{2}} x \right) - 2^n f \left( 2^\frac{m}{2} x \right) \right\| = 2^n \left\| 2^m f \left( 2^\frac{m}{2} \cdot 2^\frac{n}{2} x \right) - f \left( 2^\frac{m}{2} x \right) \right\|
\leq \frac{2}{3} \sum_{i=0}^{m-1} 2^{i+n} A \left( 2^{\frac{i+n}{2}} x, 2^{\frac{i+n}{2}} x, 2^{\frac{i+n}{2}} x \right)
\rightarrow 0 \text{ as } n \rightarrow \infty
\] (24)
for all $x \in E$. Since $F$ is complete, there exists a mapping $Q(x)$ such that
\[
Q(x) = \lim_{m \rightarrow \infty} 2^m f \left( 2^{\frac{m}{2}} x \right)
\]
for all $x \in E$. Replacing $(x, y, z)$ by $(2^\frac{m}{2} x, 2^\frac{m}{2} y, 2^\frac{m}{2} z)$ in (15) and multiply by $2^m$, we arrive
\[
2^m \left\| f \left( \sqrt{2} x \right) + f \left( \sqrt{2} y \right) + f \left( \sqrt{2} z \right) - \frac{f(2^\frac{m}{2} x) f(2^\frac{m}{2} y)}{f(2^\frac{m}{2} x) + f(2^\frac{m}{2} y)} - \frac{f(2^\frac{m}{2} y) f(2^\frac{m}{2} z)}{f(2^\frac{m}{2} y) + f(2^\frac{m}{2} z)} - \frac{f(2^\frac{m}{2} z) f(2^\frac{m}{2} x)}{f(2^\frac{m}{2} z) + f(2^\frac{m}{2} x)} \right\| \leq 2^m A(2^\frac{m}{2} x, 2^\frac{m}{2} y, 2^\frac{m}{2} z)
\]
for all $x, y, z \in E$. Letting $m$ tends to infinity in the above inequality we see that $Q(x)$ satisfies the radical reciprocal functional equation (8) for all $x, y, z \in E$. To prove $Q(x)$ is unique, let $Q'(x)$ be another radical reciprocal quadratic functional equation satisfying (8) and (17) such that $Q(2^\frac{m}{2} x) = 2^m Q(x)$ and $Q'(2^\frac{m}{2} x) = 2^m Q'(x)$ for all $x \in E$. Now,
\[
\left\| Q(x) - Q'(x) \right\| = \frac{1}{2^m} \left\| Q(2^\frac{m}{2} x) - Q'(2^\frac{m}{2} x) \right\|
\leq \frac{1}{2^m} \left\{ \left\| Q(2^\frac{m}{2} x) - f(2^\frac{m}{2} x) \right\| + \left\| f(2^\frac{m}{2} x) - Q'(2^\frac{m}{2} x) \right\| \right\}
\leq \frac{4}{3} \sum_{i=0}^{\infty} 2^i A \left( 2^\frac{i+m}{2} x, 2^\frac{i+m}{2} x, 2^\frac{i+m}{2} x \right)
\rightarrow 0 \text{ as } n \rightarrow \infty
for all \( x \in E \). Thus \( Q(x) \) is unique. Hence theorem holds for \( j = 1 \). Setting \( x \) by \( \frac{\eta}{2} \) in (19), we get

\[
\left\| 3f(x) - \frac{3}{2}f \left( \frac{x}{\sqrt{2}} \right) \right\| \leq A \left( \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}} \right)
\]

for all \( x \in E \). It follows from (25), we have

\[
\left\| f(x) - \frac{1}{2}f \left( \frac{x}{\sqrt{2}} \right) \right\| \leq \frac{1}{3}A \left( \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}} \right)
\]

for all \( x \in E \). Replacing \( x \) by \( \frac{\eta}{2} \) and divided by \( \frac{\eta}{2} \) in (26), we obtain

\[
\left\| \frac{1}{2}f \left( \frac{x}{\sqrt{2}} \right) - \frac{1}{2^2}f \left( \frac{x}{(\sqrt{2})^2} \right) \right\| \leq \frac{1}{3} \cdot 2 \cdot A \left( \frac{x}{(\sqrt{2})^2}, \frac{x}{(\sqrt{2})^2}, \frac{x}{(\sqrt{2})^2} \right)
\]

for all \( x \in E \). The rest of the proof is similar to that of case \( j = 1 \). Thus the theorem holds for \( j = -1 \). Hence the proof is complete.

**Example 4.2.** Let \( f : E \rightarrow F \) be a mapping fulfilling the inequality \( \|Df(x, y, z)\| \leq \eta \), where \( \eta > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{|2\eta|}{3}
\]

for all \( x \in E \).

**Corollary 4.3.** Let \( f : E \rightarrow F \) be a mapping fulfilling the inequality \( \|Df(x, y, z)\| \leq \eta(\||x|^a + ||y||^a + ||z||^a) \), where \( \eta > 0, a > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{2\eta||x||^a}{1 - 2^a + 1}, \quad a \neq 2
\]

for all \( x \in E \).

**Corollary 4.4.** Let \( f : E \rightarrow F \) be a mapping fulfilling the inequality \( \|Df(x, y, z)\| \leq \eta(\||x|^a + ||y||^b + ||z||^c) \), where \( \eta > 0, a, b, c > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{2\eta||x||^a}{3(1 - 2^a + 1)} + \frac{2\eta||x||^b}{3(1 - 2^b + 1)} + \frac{2\eta||x||^c}{1 - 2^c + 1}, \quad a, b, c \neq 2
\]

for all \( x \in E \).

**Corollary 4.5.** Let \( f : E \rightarrow F \) be a mapping fulfilling the inequality \( \|Df(x, y, z)\| \leq \eta||x||^a||y||^b||z||^c \), where \( \eta > 0, a > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{2\eta||x||^a}{3(1 - 2^a + 1)}, \quad 3a \neq 2
\]

for all \( x \in E \).
Corollary 4.6. Let \( f : E \to F \) be a mapping fulfilling the inequality \( \|Df(x, y, z)\| \leq \eta \|x\|^a \|y\|^b \|z\|^c \), where \( \eta > 0 \), \( a, b, c > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{2\eta \|x\|^{a+b+c}}{3 \left| 1 - \frac{a+b+c}{2} \right|}, \quad a + b + c \neq 2
\]

for all \( x \in E \).

Corollary 4.7. Let \( f : E \to F \) be a mapping fulfilling the inequality

\[
\|Df(x, y, z)\| \leq \eta \left( \|x\|^{3a} + \|y\|^{3a} + \|z\|^{3a} + \|x\|^a \|y\|^b \|z\|^c \right),
\]

where \( \eta > 0 \), \( a > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{8\eta \|x\|^{3a}}{3 \left| 1 - \frac{3a}{2} \right|}, \quad 3a \neq 2
\]

for all \( x \in E \).

Corollary 4.8. Let \( f : E \to F \) be a mapping fulfilling the inequality

\[
\|Df(x, y, z)\| \leq \eta \left( \|x\|^{a+b+c} + \|y\|^{a+b+c} + \|z\|^{a+b+c} + \|x\|^a \|y\|^b \|z\|^c \right),
\]

where \( \eta > 0 \), \( a, b, c > 0 \) and for all \( x, y, z \in E \). Then there exists a unique radical reciprocal quadratic function satisfying the functional equation (8) and

\[
\|f(x) - Q(x)\| \leq \frac{8\eta \|x\|^{a+b+c}}{3 \left| 1 - \frac{a+b+c}{2} \right|}, \quad a + b + c \neq 2
\]

for all \( x \in E \).

References

On Mildly B-Normal Spaces and Some Functions

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Abstract: In this paper, by using Bg-closed sets we obtain a characterization of mildly B-normal spaces and use it to improve the preservation theorems of mildly B-normal spaces.

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Keywords: Bg-closed sets, characterization of mildly B-normal spaces, mildly B-normal spaces.

1. Introduction and Preliminaries

The notion of mildly normal spaces was introduced by Singal and Singal [14]. Palaniappan and Rao [12] have defined and investigated the notion of regular g-closed sets as a generalization of g-closed sets due to Levine [6]. In this paper, by using regular Bg-closed sets we obtain a characterization of mildly B-normal simply extended topological spaces.

Throughout this paper, \((X, \tau(B_X)), (Y, \sigma(B_Y))\) and \((Z, \eta(B_Z))\) (briefly X, Y and Z) will denote simply extended topological spaces.

Definition 1.1. A subset A of a topological space X is said to be

(1) regular open [5] if \(A = \text{int}(\text{cl}(A))\);

(2) regular g-closed (briefly rg-closed) [12] if \(\text{cl}(A) \subset U\) whenever \(A \subset U\) and \(U\) is a regular open set in X.

(3) generalized closed (briefly g-closed) [6] if \(\text{cl}(A) \subset U\) whenever \(A \subset U\) and \(U\) is open in X.

(4) rg-open (resp. g-open, regular closed) if the complement of A is rg-closed (resp. g-closed, regular open). The family of all regular open (resp. regular closed) sets of X is denoted by RO(X) (resp. RC(X)).

Definition 1.2 ([15]). A topological space X is said to be mildly normal if for every pair of disjoint \(H, K \in RC(X)\), there exist disjoint open sets \(U, V\) of X such that \(H \subset U\) and \(K \subset V\).

Definition 1.3 ([12]). A subset A of X is said to be quasi H-closed relative to X, if for every cover \(\{V_\alpha : \alpha \in \nabla\}\) of A by open sets of X, there exists a finite subset \(\nabla_0\) of \(\nabla\) such that \(A \subset \cup\{\text{cl}(V_\alpha) : \alpha \in \nabla_0\}\).

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Definition 1.4 (5). A subset $a$ of a space $X$ is said to be $\alpha$-regular if for each point of $x \in A$ and each open set $U$ of $X$ containing $x$, there exists an open set $G$ of $X$ such that $x \in G \subseteq \text{cl}(G) \subseteq U$.

Definition 1.5 (13). A subset $a$ of a topological space $X$ is said to be $\alpha$-paracompact if every cover of $A$ by open sets of $X$ is defined by a cover of $A$ which consists of open sets of $X$ and is locally finite in $X$.

Definition 1.6 (14). A topological space $X$ is said to be mildly-normal if for every pair of disjoint $H, K \in \text{RC}(X)$, there exist disjoint open sets $U, V$ of $X$ such that $H \subseteq U$ and $K \subseteq V$.

Definition 1.7 (10). A function $f : X \rightarrow Y$ is said to be almost $g$-continuous (resp. almost $rg$-continuous) if $f^{-1}(R)$ is $g$-closed (resp. $rg$-closed) in $X$, for every $R \in \text{RC}(Y)$.

Definition 1.8. A function $f : X \rightarrow Y$ is said to be

1. $g$-continuous [3] (resp. $rg$-continuous [12]) if $f^{-1}(F)$ is $g$-closed (resp. $rg$-closed) in $X$ for every closed set $F$ of $Y$;
2. $R$-map [4], $rc$-continuous [4] or regular irresolute [12] (resp. almost continuous [14]) if $f^{-1}(V) \in \text{RO}(X)$ (resp. $\tau(X)$) for every $V \in \text{RO}(Y)$;
3. completely continuous [1] or regular continuous [12] if $f^{-1}(V) \in \text{RO}(X)$ for every $V \in \text{RO}(Y)$.

Definition 1.9 (10). A topological space $X$ is said to be regular-$T_{1/2}$ if every $rg$-closed set of $X$ is regular closed.

Definition 1.10 (12). A function $f : X \rightarrow Y$ is said to be $rg$-irresolute if $f^{-1}(F)$ is $rg$-closed in $X$ for every $rg$-closed set $F$ of $Y$.

Definition 1.11. A function $f : X \rightarrow Y$ is said to be

1. regular closed [12] (resp. $g$-closed [8], $rg$-closed [10]) if $f(F)$ is regular closed (resp. $g$-closed, $rg$-closed [10]) in $Y$ for every closed set $F$ of $X$;
2. $rc$-preserving [10] (resp. almost closed [14], almost $g$-closed [10], almost $rg$-closed [10]) if $f(F)$ is regular closed (resp. closed, $g$-closed, $rg$-closed) in $Y$ for every $F \in \text{RC}(X)$.

Remar 1.12 (11). In among others, it is shown that a compact set of a regular space is $rg$-closed.

Definition 1.13 (7). Levine in 1964 defined $\tau(B) = \{O \cup (\hat{O} \cap B) : O, \hat{O} \in \tau\}$ and called it simple extension of $\tau$ by $B$, where $B \notin \tau$. The sets in $\tau(B)$ are called $B$-open sets. and the complement of $B$-open set is called $B$-closed.

Definition 1.14 (7). Let $S$ be a subset of a simply extended topological space $X$. Then

1. The $B$-closure of $S$, denoted by $Bcl(S)$, is defined as $\cap \{F : S \subseteq F \text{ and } F \text{ is } B \text{-closed}\}$;
2. The $B$-interior of $S$, denoted by $Bint(S)$, is defined as $\cup \{F : F \subseteq S \text{ and } F \text{ is } B \text{-open}\}$.

Definition 1.15. A subset $A$ of a simply extended topological space $(X, \tau(B_X))$ is called $Bg$-closed set [2] if $Bcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$. The complement of $Bg$-closed set is called $Bg$-open set.

Definition 1.16 (9). A function $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ is called $B$-continuous if $f^{-1}(V)$ is $B$-open in $X$, for every $B$-open set $V$ of $Y$. 

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2. Regular Bg-closed Sets

Definition 2.1. A subset $A$ is said to be regular B-open (resp. regular B-closed) if $A = \text{Bint}(\text{Bcl}(A))$ (resp. $A = \text{Bcl}(\text{Bint}(A))$).

The family of regular B-open (resp. regular B-closed) sets of a simply extended topological space $X$ is denoted by $\text{BRO}(X)$ (resp. $\text{BRC}(X)$).

Definition 2.2. A subset $A$ of a simply extended topological space $X$ is said to be

(1). regular Bg-closed (briefly rBg-closed) if $\text{Bcl}(A) \subset U$ whenever $A \subset U$ and $U \in \text{BRO}(X)$.

(2). B-generalized closed (briefly Bg-closed) if $\text{Bcl}(A) \subset U$ whenever $A \subset U$ and $U$ is B-open in $X$.

(3). rBg-open (resp. Bg-open) if the complement of $A$ is rBg-closed (resp. Bg-closed).

Result 2.3. We have the following implications for properties of subsets:

regular B-closed $\Rightarrow$ B-closed $\Rightarrow$ Bg-closed $\Rightarrow$ rBg-closed.

where none of these implications is reversible as shown by Examples (below).

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset\}$ and $B = \{b, c\}$ then $\tau(B) = \{\emptyset, X, \{b, c\}\}$. Then

(1). $\{a, b\}$ is Bg-closed but not B-closed.

(2). $\{b\}$ is Brg-closed but not Bg-closed.

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $B = \{b\}$ then $\tau(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is B-closed but not regular B-closed.

3. Characterization of Mildly B-normal Spaces

Definition 3.1. A simply extended topological space $X$ is said to be mildly B-normal if for every pair of disjoint $H, K \in \text{BRC}(X)$, there exist disjoint B-open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$.

Lemma 3.2. A subset $A$ of a simply extended topological space $X$ is rBg-open if and only if $F \subset \text{Bint}(A)$ whenever $F \in \text{BRC}(X)$ and $F \subset A$.

Theorem 3.3. The following are equivalent for a simply extended topological space $X$.

(1). $X$ is mildly B-normal;

(2). for any disjoint $H, K \in \text{BRC}(X)$, there exist disjoint Bg-open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$;

(3). for any disjoint $H, K \in \text{BRC}(X)$, there exist disjoint rBg-open sets $U, V$ of $X$ such that $H \subset U$ and $K \subset V$;

(4). for any disjoint $H \in \text{BRC}(X)$ and any $V \in \text{BRO}(X)$ containing $H$, there exists a rBg-open set $U$ of $X$ such that $H \subset U \subset Bcl(U) \subset V$.

Proof. It is obvious that (1) implies (2) and (2) implies (3).

(3) $\Rightarrow$ (4) Let $H \in \text{BRC}(X)$ and $H \subset V \in \text{BRO}(X)$. There exist disjoint rBg-open sets $U, W$ such that $H \subset U$ and $X - V \subset W$. By Lemma 3.2, we have $X - V \subset \text{Bint}(W)$ and $U \cap \text{Bint}(W) = \emptyset$. Therefore, we obtain $Bcl(U) \cap Bint(W) = \emptyset$ and hence $H \subset U \subset Bcl(U) \subset X - Bint(W) \subset V$.

(4) $\Rightarrow$ (1) Let $H, K$ be disjoint regular B-closed sets of $X$. Then $H \subset X - K \in \text{BRO}(X)$ and there exists a rBg-open set $G$ of $X$ such that $H \subset G \subset Bcl(G) \subset X - K$. Put $U = \text{Bint}(G)$ and $V = X - Bcl(G)$. Then $U$ and $V$ are disjoint B-open sets of $X$ such that $H \subset U$ and $K \subset V$. Therefore, $X$ is mildly B-normal.
4. Some Functions

**Definition 4.1.** A function \( f : X \rightarrow Y \) is said to be almost Bg-continuous (resp. almost rBg-continuous) if \( f^{-1}(R) \) is Bg-closed (resp. rBg-closed) for every \( R \in \text{BG}(Y) \).

**Definition 4.2.** A function \( f : X \rightarrow Y \) is said to be

1. Bg-continuous (resp. rBg-continuous) if \( f^{-1}(F) \) is Bg-closed (resp. rBg-closed) for every B-closed set \( F \) of \( Y \);
2. BR-map (resp. almost B-continuous) if \( f^{-1}(V) \in \text{BR}(X) \) (resp. \( \tau(B)(X) \)) for every \( V \in \text{BR}(Y) \);
3. completely B-continuous if \( f^{-1}(V) \in \text{BR}(X) \) for every B-open set \( V \) of \( Y \).

From the definitions stated above, we obtain the following diagram:

\[
\begin{array}{ccc}
\text{complete B-continuity} & \rightarrow & \text{BR-map} \\
\downarrow & & \downarrow \\
\text{B-continuity} & \rightarrow & \text{almost B-continuity} \\
\downarrow & & \downarrow \\
\text{Bg-continuity} & \rightarrow & \text{almost Bg-continuity} \\
\downarrow & & \downarrow \\
\text{rBg-continuity} & \rightarrow & \text{almost rBg-continuity}
\end{array}
\]

**Remark 4.3.** None of the implications in Diagram I is reversible as shown by the following Examples.

**Example 4.4.**

1. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X\} \) and \( B_X = \{a\} \) then \( \tau(B_X) = \{\phi, X, \{a\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a, b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a, b\}\} \). Let \( f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y)) \) be an identity map. Then \( f \) is BR-map (resp. almost B-continuous) but not completely B-continuous (resp. B-continuous).

2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( B_X = \{a, b\} \) then \( \tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y)) \) be an identity map. Then \( f \) is almost Bg-continuous but not Bg-continuous.

**Example 4.5.**

1. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X\} \) and \( B_X = \{a\} \) then \( \tau(B_X) = \{\phi, X, \{a\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{a\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y)) \) be an identity map. Then \( f \) is B-continuous but not completely B-continuous.

2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a, b\}\} \) and \( B_X = \{b\} \) then \( \tau(B_X) = \{\phi, X, \{a, b\}\} \). Let \( \sigma = \{\phi, Y, \{a, b\}\} \) and \( B_Y = \{b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a, b\}\} \). Let \( f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y)) \) be an identity map. Then \( f \) is almost B-continuous.

**Example 4.6.** Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}\} \) and \( B_X = \{a, b\} \) then \( \tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\} \). Let \( \sigma = \{\phi, Y\} \) and \( B_Y = \{b\} \) then \( \sigma(B_Y) = \{\phi, Y, \{a\}, \{b\}\} \). Let \( f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y)) \) be an identity map. Then \( f \) is Bg-continuous (resp. almost B-continuous) but not B-continuous (resp. almost Bg-continuous).
Example 4.7. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{a, b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{a\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_X))$ be an identity map. Then $f$ is rBg-continuous but not Bg-continuous.

Example 4.8. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{a\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{a, b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_X))$ be an identity map. Then $f$ is almost rBg-continuous but neither almost Bg-continuous nor rBg-continuous.

Definition 4.9. A simply extended topological space $X$ is said to be regular $B$-$T_{1/2}$ if every rBg-closed set of $X$ is regular $B$-closed.

Proposition 4.10. If a function $f : X \rightarrow Y$ is rBg-continuous and $X$ is regular B-$T_{1/2}$, then $f$ is completely B-continuous.

Proof. Let $F$ be any B-closed set of $Y$. Since $f$ is rBg-continuous, $f^{-1}(F)$ is rBg-closed in $X$ and hence $f^{-1}(F) \in BRC(X)$. Therefore, $f$ is completely B-continuous.

Definition 4.11. A function $f : X \rightarrow Y$ is said to be rBg-irresolute if $f^{-1}(F)$ is rBg-closed in $X$ for every rBg-closed set $F$ of $Y$. Every rBg-irresolute function is rBg-continuous but not conversely as shown by the following Example.

Example 4.12. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{a, b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{a\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_X))$ be an identity map. Then $f$ is B-continuous and Bg-closed but not rBg-irresolute.

Corollary 4.13. If $f : X \rightarrow Y$ is rBg-irresolute and $X$ is regular $B$-$T_{1/2}$, then $f$ is BR-map.

Definition 4.14. A function $f : X \rightarrow Y$ is said to be

1. regular B-closed (resp. Bg-closed, rBg-closed) if $f(F)$ is regular B-closed (resp. Bg-closed, rBg-closed) in $Y$ for every B-closed set $F$ of $X$;

2. rBc-preserving (resp. almost B-closed, almost Bg-closed, almost rBg-closed) if $f(F)$ is regular B-closed (resp. B-closed, Bg-closed, rBg-closed) in $Y$ for every $F \in BRC(X)$.

From the definitions stated above, we obtain the following diagram:

```
regular B-closed → rBc-preserving
  ↓    ↓
B-closed → almost B-closed
  ↓    ↓
Bg-closed → almost Bg-closed
  ↓    ↓
rBg-closed → almost rBg-closed
```

Remark 4.15. None of the implications in Diagram II is reversible.

Example 4.16. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{b\}\}$. Let $\sigma = \{\phi, Y, \{a\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a\}, \{b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_X))$ be an identity map. Then $f$ is

1. rBc-preserving but not regular B-closed.
(2). regular $B$-closed but not $B$-closed.

Example 4.17.

(1). Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X\}$ and $B_X = \{a\}$ then $\tau(B_X) = \{\phi, X, \{a\}\}$. Let $\sigma = \{\phi, Y\}$ and $B_Y = \{a, b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a, b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then $f$ is $B$-closed but not almost $B$-closed.

(2). Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X\}$ and $B_X = \{b, c\}$ then $\tau(B_X) = \{\phi, X, \{b, c\}\}$. Let $\sigma = \{\phi, Y\}$ and $B_Y = \{a, b\}$ then $\sigma(B_Y) = \{\phi, Y, \{a, b\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then $f$ is $Bg$-closed but not $B$-closed.

(3). Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $B_X = \{b\}$ then $\tau(B_X) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $B_Y = \{b\}$ then $\sigma(B_Y) = \{\phi, Y, \{b\}, \{c\}, \{b, c\}\}$. Let $f : (X, \tau(B_X)) \rightarrow (Y, \sigma(B_Y))$ be an identity map. Then $f$ is almost $Bg$-closed (resp. $Bg$-closed, $Bg$-closed) but not almost $B$-closed (resp. almost $Bg$-closed, $rB$g-closed).

Proposition 4.18. Let $X$ and $Y$ be simply extended topological spaces. Let $f : X \rightarrow Y$ be a function. Then

(1). if $f$ is $rBg$-continuous $rBc$-preserving, then it is $rBg$-irresolute;

(2). if $f$ is an $BR$-map and $rBg$-closed, then $f(A)$ is $rBg$-closed in $Y$ for every $rBg$-closed set $A$ of $X$.

Proof.

(1). Let $A$ be any $rBg$-closed set of $Y$ and $U \in BRO(X)$ containing $f^{-1}(A)$. Put $V = Y - f(X - U)$, then we have $A \subset V$, $f^{-1}(V) \subset U$ and $V \in BRO(Y)$ since $f$ is $rBc$-preserving. Hence we obtain $Bcl(A) \subset V$ and hence $f^{-1}(Bcl(A)) \subset U$. By the $rBg$-continuity of $f$, we have $Bcl(f^{-1}(A)) \subset Bcl(f^{-1}(Bcl(A))) \subset U$. This shows that $f^{-1}(A)$ is $rBg$-closed in $X$. Therefore, $f$ is $rBg$-irresolute.

(2). Let $A$ be any $rBg$-closed set of $X$ and $V \in BRO(X)$ containing $f(A)$. Since $f$ is an $BR$-map, $f^{-1}(V) \in BRO(X)$ and $A \subset f^{-1}(V)$. Therefore, we have $Bcl(A) \subset f^{-1}(V)$ and hence $f(Bcl(A)) \subset V$. Since $f$ is $rBg$-closed, $f(Bcl(A))$ is $rBg$-closed in $Y$ and hence we obtain $Bcl(f(A)) \subset Bcl(f(Bcl(A))) \subset U$. This shows that $f(A)$ is $rBg$-closed in $Y$.

Corollary 4.19. Let $X$ and $Y$ be simply extended topological spaces. Let $f : X \rightarrow Y$ be a function. Then

(1). if $f$ is $B$-continuous regular $B$-closed, $f^{-1}(A)$ is $rBg$-closed in $X$ for every $rBg$-closed set $A$ of $Y$;

(2). if $f$ is $BR$-map and $B$-closed, $f(A)$ is $rBg$-closed in $Y$ for every $rBg$-closed set $A$ if $X$.

Proposition 4.20. Let $X$ and $Y$ be simply extended topological spaces. A surjection $f : X \rightarrow Y$ is almost $rBg$-closed (resp. almost $B$g-closed) if and only if for each subset $S$ of $Y$ and each $U \in BRO(X)$ containing $f^{-1}(S)$ there exists an $rBg$-open (resp. $B$g-open) set $V$ of $Y$ such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. We prove only the first case, the proof of the second being entirely analogous.

Necessity: Suppose that $f$ is almost $rBg$-closed. Let $S$ be a subset of $Y$ and $U \in BRO(X)$ containing $f^{-1}(S)$. Put $V = Y - f(X - U)$, then $V$ is an $rBg$-open set of $Y$ such that $S \subset V$ and $f^{-1}(V) \subset U$. 


Sufficiency: Let $F$ be any regular $B$-closed set of $X$. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in \text{BRO}(X)$. There exists an $rBg$-open set $V$ of $Y$ such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \cap Y - V$ and $F \subset f^{-1}(Y - V)$. Hence, we obtain $f(F) = Y - V$ and $f(F)$ is $rBg$-closed in $Y$. This shows that $f$ is almost $rBg$-closed.

5. Preservation Theorems

In this section we investigate preservation theorems concerning mildly $B$-normal spaces.

Theorem 5.1. Let $X$ and $Y$ be simply extended topological spaces. If $f : X \rightarrow Y$ is an almost $rBg$-continuous $rBc$-preserving (resp. almost $B$-closed) injection and $Y$ is mildly $B$-normal (resp. $B$-normal), then $X$ is mildly $B$-normal.

Proof. Let $A$ and $C$ be any disjoint regular $B$-closed sets of $X$. Since $f$ is an $rBc$-preserving (resp. almost $B$-closed) injection, $f(A)$ and $f(C)$ are disjoint regular $B$-closed (resp. $B$-closed) sets of $Y$. By the mild $B$-normality (resp. $B$-normality) of $Y$, there exist disjoint $B$-open sets $U$ and $V$ of $X$ such that $f(A) \subset U$ and $f(C) \subset V$. Now, put $G = \text{Bint}(\text{Bcl}(U))$ and $H = \text{Bint}(\text{Bcl}(V))$, then $G$ and $H$ are disjoint regular $B$-open sets such that $f(A) \subset G$ and $f(C) \subset H$. Since $f$ is almost $rBg$-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint $rBg$-open sets containing $A$ and $C$, respectively. It follows from Theorem 3.3 that $X$ is mildly $B$-normal.

Theorem 5.2. Let $X$ and $Y$ be simply extended topological spaces. If $f : X \rightarrow Y$ is a completely $B$-continuous almost $B$-closed surjection and $X$ is mildly $B$-normal, then $Y$ is $B$-normal.

Proof. Let $A$ and $C$ be any disjoint $B$-closed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(C)$ are disjoint regular $B$-closed sets of $X$. Since $X$ is mildly $B$-normal, there exist disjoint $B$-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subset U$ and $f^{-1}(C) \subset V$. Let $G = \text{Bint}(\text{Bcl}(U))$ and $H = \text{Bint}(\text{Bcl}(V))$, then $G$ and $H$ are disjoint regular $B$-open sets such that $f^{-1}(A) \subset G$ and $f^{-1}(C) \subset H$. By Proposition 4.20, there exists $Bg$-open sets $K$ and $L$ of $Y$ such that $A \subset K$, $C \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since $G$ and $H$ are disjoint, so are $K$ and $L$. Since $K$ and $L$ are $Bg$-open, we obtain $A \subset \text{Bint}(K)$, $C \subset \text{Bint}(L)$ and $\text{Bint}(K) \cap \text{Bint}(L) = \emptyset$. This shows that $Y$ is $B$-normal.

Corollary 5.3. Let $X$ and $Y$ be simply extended topological spaces. If $f : X \rightarrow Y$ is a completely $B$-continuous $B$-closed surjection and $X$ is mildly $B$-normal, then $Y$ is $B$-normal.

Theorem 5.4. Let $X$ and $Y$ be simply extended topological spaces. Let $f : X \rightarrow Y$ be an $BR$-map (resp. almost $B$-continuous) and almost $rBg$-closed surjection. If $X$ is mildly $B$-normal (resp. $B$-normal), then $Y$ is mildly $B$-normal.

Proof. Let $A$ and $C$ be any disjoint regular $B$-closed sets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(C)$ are disjoint regular $B$-closed (resp. B-closed) sets of $X$. Since $X$ is mildly $B$-normal (resp. $B$-normal), there exist disjoint $B$-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subset U$ and $f^{-1}(C) \subset V$. Put $G = \text{Bint}(\text{Bcl}(U))$ and $H = \text{Bint}(\text{Bcl}(V))$, then $G$ and $H$ are disjoint regular $B$-open sets of $X$ such that $f^{-1}(A) \subset G$ and $f^{-1}(C) \subset H$. By Proposition 4.20, there exists $rBg$-open sets $K$ and $L$ of $Y$ such that $A \subset K$, $C \subset L$, $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since $G$ and $H$ are disjoint, so are $K$ and $L$. It follows from Theorem 3.3 that $Y$ is mildly $B$-normal.

Corollary 5.5. Let $X$ and $Y$ be simply extended topological spaces. If $f : X \rightarrow Y$ is an almost $B$-continuous almost $B$-closed surjection and $X$ is $B$-normal, then $Y$ is mildly $B$-normal.
References

Abstract: A ring $R$ is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$. It is proved that a ring $R$ is strongly symmetric if and only if its polynomial ring $R[x]$ is strongly symmetric if and only if its Laurent polynomial ring $R[x, x^{-1}]$ is strongly symmetric. We also show that for a right Ore ring $R$ with $Q$ its classical right quotient ring, $R$ is strongly symmetric if and only if $Q$ is strongly symmetric. Finally we proved that, let $R$ be an algebra over a commutative ring $S$, and $D$ be the Dorroh extension of $R$ by $S$. If $R$ is strongly symmetric and $S$ is a domain, then $D$ is strongly symmetric.

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1. Introduction

Throughout this note all rings are associative with identity unless otherwise stated. According to Lambek [6], a ring $R$ is called symmetric if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$; while Anderson and Camillo [3] took the term $ZC_3$ for this notion. Lambek proved that a ring $R$ is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with $n$ any positive integer, implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation $\sigma$ of the set $\{1, 2, \ldots, n\}$ and $r_i \in R$ [6, Proposition 1]. Anderson and Camillo obtained this result independently in [3, Theorem I.1]. Given a ring $R$, $\lambda R(-)(\lambda R(-))$ is used for the right (left) annihilator in $R$. According to Cohn [8], a ring $R$ is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [3], observing the rings whose zero products commute, used the term $ZC_2$ for what is called reversible, and Krempa-Niewieczeral [5] took the term $C_0$ for it. It is obvious that commutative rings are symmetric and symmetric rings are reversible; but reversible rings need not be symmetric and symmetric rings need not be commutative by the results of Anderson and Camillo [3, Examples I.5 and II.5] and Marks [4, Examples 5 and 7]. A ring is called reduced if it has no nonzero nilpotent elements. Reduced rings are symmetric by the result of Anderson and Camillo [3, Theorem I.3], but there are many nonreduced commutative (so symmetric) rings. Another generalization of a reduced ring is an Armendariz ring. Rege and Chhawchharia [7] called a ring $R$ Armendairz if whenever any polynomials $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each $i$ and $j$. Huh and et al. [1, Example 3.1], showed that polynomial rings over symmetric rings need not be symmetric. In the paper, we consider these symmetric rings over which polynomial rings are symmetric and call them strongly symmetric, i.e., a ring $R$ is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$.

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2. Strongly Symmetric Rings

Definition 2.1. A ring $R$ is called strongly symmetric, if whenever polynomials $f(x), g(x), h(x)$ in $R[x]$ satisfy $f(x)g(x)h(x) = 0$, then $f(x)h(x)g(x) = 0$.

Clearly, every strongly symmetric ring is symmetric. But the converse is not true [1, Example 3.1]. It is obvious that any reduced rings are strongly symmetric and symmetric.

Lemma 2.2. The class of strongly symmetric rings is closed under subrings (not necessarily with identity) and direct products.

Recall that an element $u$ of a ring $R$ is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Proposition 2.3. Let $\Delta$ be a multiplicatively closed subset of a ring $R$ consisting of central regular elements. Then $R$ is strongly symmetric ring if and only if so is $\Delta^{-1}R$.

Proof. It is enough to show that the necessity. Suppose that $R$ is strongly symmetric. Let $\phi \varphi \psi = 0$, with $\phi = u^{-1}f(x), \varphi = v^{-1}g(x), \psi = w^{-1}h(x), u, v, w \in \Delta$ and $f(x), g(x), h(x) \in R[x]$. Since $\Delta$ is contained in the center of $R$, we have $0 = \phi \varphi \psi = u^{-1}f(x)v^{-1}g(x)w^{-1}h(x) = (u^{-1}v^{-1}w^{-1})f(x)g(x)h(x)$ and so $f(x)g(x)h(x) = 0$. But $R$ is strongly symmetric by the condition, so $f(x)h(x)g(x) = 0$ and $\phi \varphi \psi = u^{-1}f(x)w^{-1}h(x)v^{-1}g(x) = (uvw)^{-1}f(x)h(x)g(x) = 0$. Hence $\Delta^{-1}R$ is strongly symmetric. □

The ring of Laurent polynomials in $x$, with coefficients in a ring $R$, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and $k, n$ are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 2.4. Let $R$ be a ring. Then $R[x]$ is strongly symmetric rings if and only if $R[x; x^{-1}]$ is strongly symmetric.

Proof. Let $\Delta = \{1, x, x^2, \ldots \}$. Then clearly $\Delta$ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$, it follows that $R[x; x^{-1}]$ is strongly symmetric by the Proposition 2.3. □

Theorem 2.5. A ring $R$ is strongly symmetric if and only if $R[x]$ is strongly symmetric.

Proof. $(\Rightarrow)$ By Lemma 2.2.

$(\Rightarrow)$ Let $f(y) = f_0 + f_1 y + \cdots + f_p y^p$, $g(y) = g_0 + g_1 y + \cdots + g_q y^q$, $h(y) = h_0 + h_1 y + \cdots + h_l y^l \in R[y]$ satisfy $f(y)g(y)h(y) = 0$, where $f_i = \sum_{s=0}^{n_i} a_s^{(i)} x^s, g_j = \sum_{t=0}^{n_j} b_t^{(j)} x^t, h_k = \sum_{u=0}^{n_k} c_u^{(k)} x^u \in R[x]$ for $i = 0, 1, \ldots, p, j = 0, 1, \ldots, q, k = 0, 1, \ldots, l$. Let $w = \deg(f_0) + \deg(f_1) + \cdots + \deg(f_p) + \deg(g_0) + \deg(g_1) + \cdots + \deg(g_q) + \deg(h_0) + \deg(h_1) + \cdots + \deg(h_l)$, where degree is as polynomials in $x$ and the degree of the zero polynomial is taken to be 0. Then $f(x^w) = f_0 + f_1 x^w + \cdots + f_p x^{pw}, g(x^w) = g_0 + g_1 x^w + \cdots + g_q x^{qw}, h(x^w) = h_0 + h_1 x^w + \cdots + h_l x^{lw} \in R[x]$ and the set of coefficients of $f_i, g_j, h_k$ (resp. $h_i k$) equals the set of coefficients of $f(x^w), g(x^w)$ (resp. $h(x^w)$). Since $f(y)g(y)h(y) = 0$ and $x$ commutes with elements of $R$, we have that $f(x^w)g(x^w)h(x^w) = 0$, thus $f(x^w)h(x^w)g(x^w) = 0 = f(y)h(y)g(y)$ since $R$ is strongly symmetric, which implies $R[x]$ is strongly symmetric. □

Corollary 2.6. Let $R$ be a strongly symmetric ring and $\{x_1, \ldots, x_n\}$ any set of commuting indeterminates over $R$. Then any subring of $R[\{x_1, \ldots, x_n\}]$ is strongly symmetric.

Proof. Let $f(y), g(y), h(y) \in R[\{x_1, \ldots, x_n\}]$ with $f(y)g(y)h(y) = 0$. Then $f(y), g(y), h(y) \in R[\{x_{x_1}, x_{x_2}, \ldots, x_{x_n}\}]$. Therefore $f(x), g(x), h(x) \in R[x_{x_1}, x_{x_2}, \ldots, x_{x_n}]$. Then $f(x)g(x)h(x) = 0$ and $\Delta^{-1}R$ is strongly symmetric. □
for some finite subset \( \{x_{a_1}, x_{a_2}, \ldots, x_{a_n}\} \subseteq \{x_i\} \). The ring \( R[[x_{a_1}, x_{a_2}, \ldots, x_{a_n}]]_y \), by induction, is strongly symmetric, so we have that \( f(y)h(y)g(y) = 0 \). Hence \( R[[x_i]] \) is strongly symmetric and thus so is any subring of \( R[[x_i]] \). \( \square \)

Let \( R \) be a ring. Suppose that \( Z(R) \) contains an infinite subring whose nonzero element are regular in \( R \), where \( Z(R) \) denotes the set of all central elements of \( R \), if \( R \) is symmetric, then \( R \) is strongly symmetric by [1, Proposition 3.3]. Another example of a strongly symmetric ring is given in the following which also shows that strongly symmetric rings are not reduced in general.

**Proposition 2.7.** Let \( R \) be a ring and \( n \) any positive integer. If \( R \) is reduced, then \( R[x]/(x^n) \) is a strongly symmetric ring, where \( (x^n) \) is the ideal generated by \( x^n \).

**Proof.** It is obvious that \( R[x]/(x^n) \) is strongly symmetric since \( R[x]/(x^n) \) is both symmetric [1, Theorem 2.3] and Armendariz [2, Theorem 5]. \( \square \)

Given a ring \( R \) and a bimodule \( R_M \), the trivial extension of \( R \) by \( M \), write \( T(R, M) \) is the ring \( R \bigoplus M \) with the usual addition and the following multiplication:

\[
(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]

Note that \( T(R, M) \) is isomorphic to the ring of all matrices \[
\begin{pmatrix}
  r & m \\
  0 & r
\end{pmatrix}
\]
where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

**Corollary 2.8.** Let \( R \) be a ring and \( T = R \bigoplus M \) be the trivial extension of \( R \) by \( R \). If \( R \) is reduced, then \( T \) is strongly symmetric.

**Proof.** \( T \cong R[x]/(x^2) \) is strongly symmetric by Proposition 2.7. \( \square \)

**Proposition 2.9.** Let \( R \) be a subdirect sum of strongly symmetric rings. Then \( R \) is strongly symmetric.

**Proof.** Let \( I_\lambda (\lambda \in \Lambda) \) be ideals of \( R \) such that \( R/I_\lambda \) is strongly symmetric and \( \cap_{\lambda \in \Lambda} I_\lambda = 0 \). Suppose that \( f(x) = \sum_{i=0}^{n} a_i x_i, g(x) = \sum_{j=0}^{m} b_j x^j, h(x) = \sum_{k=0}^{r} c_k x^k \in R[x] \) satisfy \( f(x)g(x)h(x) = 0 \). Then \( \bar{f}(x)\bar{h}(x)\bar{g}(x) = 0 \) in \( (R/I_\lambda)[x] \) for each \( \lambda \in \Lambda \) since \( R/I_\lambda \) is strongly symmetric. So \( \sum_{i+j+k=n} a_ib_kb_l \in I_\lambda \) for \( l = 0, 1, \ldots, n + m + r \) and any \( \lambda \in \Lambda \) which implies that \( \sum_{i+j+k=n} a_ib_kb_l = 0 \) for \( l = 0, 1, \ldots, n + m + r \) since \( \cap_{\lambda \in \Lambda} I_\lambda = 0 \), and we obtain \( f(x)h(x)g(x) = 0 \). \( \square \)

**Proposition 2.10.** Let \( R \) be a ring and \( I \) be a proper ideal of \( R \). If \( R/I \) is strongly symmetric and \( I \) is reduced (as a ring without identity) then \( R \) is strongly symmetric.

**Proof.** Let \( f(x), g(x), h(x) \in R[x] \) satisfy \( f(x)g(x)h(x) = 0 \). Then \( g(x)h(x)f(x) = 0 \) and \( (f(x)h(x)g(x))h(x)f(x)h(x)g(x) = 0 \Rightarrow (h(x)f(x)h(x)g(x))f(x)h(x)g(x) = 0 \Rightarrow h(x)f(x)h(x)g(x)f(x)h(x)g(x) = 0 \Rightarrow (f(x)h(x)g(x)f(x)h(x)g(x)f(x)h(x)g(x)) = 0 \Rightarrow (f(x)h(x)g(x)f(x)h(x)g(x)f(x)h(x)g(x)f(x)h(x)g(x)) = 0 \) and so we have \( 0 = (f(x)h(x)g(x)f(x)h(x)g(x)f(x)h(x)g(x))f(x)h(x)g(x) = (f(x)h(x)g(x))^3 \). Thus \( f(x)h(x)g(x) = 0 \) since \( f(x)h(x)g(x) \in I[x] \) and \( I[x] \) is reduced. Therefore \( R \) is strongly symmetric. \( \square \)

**Theorem 2.11.** Let \( R \) be a right Ore ring and \( Q \) be the classical right quotient ring of \( R \). Then \( R \) is strongly symmetric if and only if so is \( Q \).
Proof. It suffices to obtain the necessity by Lemma 2.2. Suppose that \( R \) is strongly symmetric and let \( \phi \varphi \psi = 0 \) for \( \phi = f(x)u^{-1}, \varphi = g(x)v^{-1} \) and \( \psi = h(x)w^{-1} \) in \( Q \). There exist \( g_1(x), u_1 \in R[x] \) with \( u_1 \) regular such that \( g_1(x)u_1 = u_1g_1(x) \) and \( u^{-1}g(x) = g(x)u^{-1} \), so we have \( 0 = \phi \varphi \psi = f(x)u^{-1}g(x)v^{-1}h(x)w^{-1} = f(x)g_1(x)u_1^{-1}v^{-1}h(x)w^{-1} \). Next there exist \( h_1(x), v_1 \in R \) with \( v_1 \) regular such that \( h(x)v_1 = v_1h_1(x) \) and \( v^{-1}h(x) = h_1(x)v^{-1} \) so we have \( 0 = \phi \varphi \psi = f(x)g_1(x)v_1^{-1}h_1(x)v_1^{-1}w^{-1} \).

Also there exist \( h_2(x), u_2 \in R[x] \) with \( u_1 \) regular such that \( h_1(x)u_2 = u_1h_2(x) \) and \( u_1^{-1}h_1(x) = h_2(x)u_2^{-1} \). So we have \( 0 = \phi \varphi \psi = f(x)u^{-1}g(x)v^{-1}h(x)w^{-1} = f(x)g_1(x)h_2(x)u_2^{-1}v^{-1}w^{-1} \). Hence we get \( f(x)g_1(x)h_2(x) = 0 \). In the following computation we use the condition that \( R \) is strongly symmetric: \( f(x)g_1(x)h_2(x) = 0, f(x)g_1(x)h_2(x)u = 0 \) and \( 0 = f(x)ug_1(x)h_2(x) = f(x)g_1(x)u_2h_2(x) \Rightarrow 0 = f(x)g_1(x)h_2(x)u_1 \) implies \( f(x)g_1(x)h_2(x) = 0 \) and \( 0 = f(x)g_1(x)h_2(x)u_1 = f(x)g_1(x)u_1h_2(x) = f(x)g_1(x)h(x)u_2 \Rightarrow f(x)g_1(x)h(x) = 0 \). Therefore \( \phi \varphi \psi = f(x)h_3(x)g_4(x)v_1^{-1}w_1^{-1}w_1^{-1} = 0 \), proving that \( Q \) is strongly symmetric.

Proposition 2.12. For an abelian ring \( R \). The following statements are equivalent:

(1) \( R \) is strongly symmetric rings.

(2) \( eR \) and \( (1 - e)R \) are strongly symmetric rings.

Proof. (1)\(\Rightarrow\)(2) is straightforward since subrings and direct products of strongly symmetric rings are strongly symmetric.

Let \( R \) be an algebra over a commutative ring \( S \). The Dorroh extension of \( R \) by \( S \) is the ring \( R \times S \) with operations \((r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \) and \((r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2) \), where \( r_1 \in R \) and \( s_i \in S \).

Proposition 2.13. Let \( R \) be an algebra over a commutative ring \( S \), and \( D \) be the Dorroh extension of \( R \) by \( S \). If \( R \) is strongly symmetric and \( S \) is a domain, then \( D \) is strongly symmetric.

Proof. Let \((f_1(x), g_1(x)), (f_2(x), g_2(x)), (f_3(x), g_3(x)) \in D \) with \((f_1(x), g_1(x))(f_2(x), g_2(x))(f_3(x), g_3(x)) = 0 \). Then \((f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \), so we have \((f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + g_2(x)f_1(x)f_3(x) + g_3(x)f_1(x)f_2(x) + g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \) and \((g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \). Since \( S \) is a domain, \((g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \) or \((g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \) or \((g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x), g_1(x)g_2(x)f_3(x) + g_1(x)g_3(x)f_2(x) + g_2(x)g_3(x)f_1(x)) = 0 \).
Say \( g_2(x) = 0 \), then \( f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) = 0 \) and so we have \( 0 = f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) = (f_1(x) + g_1(x))f_2(x)(f_3(x) + g_3(x)) = (f_1(x) + g_1(x))f_2(x)f_3(x) + f_1(x)g_3(x)f_2(x) + g_1(x)f_3(x)f_2(x) = f_1(x)f_3(x)f_2(x) + f_1(x)g_3(x)f_2(x) + g_1(x)f_3(x)f_2(x) + g_1(x)g_3(x)f_2(x) + f_1(x)f_3(x)g_2(x) + f_1(x)g_3(x)g_2(x) + f_1(x)g_3(x)g_2(x) + g_1(x)f_3(x)g_2(x) = 0 \) and so we have \( 0 = f_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) = (f_1(x) + g_1(x))(f_3(x) + g_3(x))f_2(x) = (f_1(x) + g_1(x))(f_2(x) + g_2(x)) = f_1(x)f_2(x)f_3(x) + f_1(x)f_2(x)(f_3(x) + g_3(x)) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)f_3(x) + g_1(x)f_2(x)(f_3(x) + g_3(x)) + g_1(x)f_2(x)g_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) + f_1(x)f_2(x)g_3(x) + g_1(x)f_2(x)g_3(x) \). Therefore we obtain \( (f_1(x), g_1(x))(f_3(x), g_3(x))(f_2(x), g_2(x)) = 0 \) in any case, proving that \( D \) is strongly symmetric.

References


A Study on the Algorithms for Equilibrium Problems on Hadamard Manifolds

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Abstract: The theory of equilibrium problems provides a unified, natural and general framework to study a wide class of problems, which arise in finance, economics, network analysis, transportation and optimization. This theory has applications across all disciplines of pure and applied sciences. Equilibrium problems include variational inequalities and related problems. The aim of this paper is to provide a survey on the algorithms for equilibrium problems that have been studied by many authors on Hadamard manifolds. This paper should be a useful reference for further research in the field of equilibrium problems and Hadamard manifolds.

MSC: 49J40, 90C33, 35A15, 47H17.

Keywords: Equilibrium problems, Mixed equilibrium problems, auxiliary principle technique, Hadamard manifold.

1. Introduction

Hadamard manifold named after Jacques Hadamard, sometimes called a Carter-Hadamard manifold after Elie Cartan is a Riemannian Manifold (M, g), that is complete and simply connected, and has everywhere non-positive sectional curvature [16]. Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. In the last decades concepts and techniques which fit in Euclidean spaces have extended to this non-linear framework. Most of the extended methods require the Riemannian manifold to have non-positive sectional curvature. This is an important property which is enjoyed by a large class of Riemannian manifolds and it is strong enough to imply tight topological restrictions and rigidity phenomena [21, 22]. Particularly, Hadamard manifolds, which are complete simply connected finite-dimensional Riemannian manifolds of non-positive sectional curvature, have turned out to be a suitable setting for diverse disciplines. Hadamard manifolds are examples of hyperbolic spaces and geodesic spaces, more precisely, a Busemain non-positive curvature space [7, 15, 23, 25].

In 2012, M.A. Noor et al. [13] gave an iterative method for solving the equilibrium problem on Hadamard Manifolds using the auxiliary principle technique. Recently, much attention has been given to study the variational inequalities, equalities, equilibrium and related optimization problems on the Riemannian manifold and Hadamard manifold. This work is useful for the development of various fields. Nemeth [26], Tang et al. [5], and Colao et al. [24] have considered the variational inequalities and equilibrium problems on Hadamard manifolds. They have studied the existence of solutions of equilibrium problems under some suitable conditions.
Let $M$ be a simply connected $m$-dimensional manifold. Given $x \in M$, the tangent space of $M$ at $x$ is denoted by $T_xM$ and the tangent bundle of $M$ by $TM = \bigcup_{x \in M} T_xM$, which is naturally a manifold. A vector field $A$ on $M$ is a mapping of $M$ into $TM$ which associates to each point $x \in M$, a vector $A(x) \in T_xM$. They assumed that $M$ can be endowed with a Riemannian metric to become a Riemannian manifold. They denoted by $(\cdot, \cdot)$ the scalar product on $T_xM$ with the associated norm $|\cdot|_x$, where the subscript $x$ will be omitted. Given a piecewise smooth curve $\gamma : [a, b] \to M$ joining $x$ to $y$ (that is, $\gamma(a) = x$ and $\gamma(b) = y$) by using the metric, we can define the length of $\gamma$ as $L(\gamma) = \int_a^b |\gamma'(t)| dt$. Then for any $x, y \in M$, the Riemannian distance $d(x, y)$, which includes the original topology on $M$, is defined by minimizing this length over the set of all such curves joining $x$ to $y$.

Let $\Delta$ be the Levi-Civita connection with $(M, (\cdot, \cdot))$. Let $\gamma$ be a smooth curve in $M$. A vector field $A$ is said to be parallel along $\gamma$ if $\Delta_{\gamma'} A = 0$. If $\gamma'$ itself is parallel along $\gamma$, he said that $\gamma$ is a geodesic and in this case $|\gamma|$ is a constant. When $|\gamma'| = 1$, $\gamma$ is said to be normalized. A geodesic joining $x$ to $y$ in $M$ is said to be minimal if its length equals $d(x, y)$.

A Riemannian manifold is complete if for any $x \in M$, all geodesics emanating from $x$ are defined for all $t \in R$. By the Hopf-Rinow theorem, we know that if $M$ is complete, then any pair of points in $M$ can be joined by a minimal geodesic. Moreover, $(M, d)$ is a complete metric space, and bounded closed subsets are compact. Let $M$ be complete. Then the exponential map $\exp_x : T_xM \to M$ at $x$ is defined by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at $x$ with velocity $v$ (i.e., $\gamma(0) = x$ and $\gamma'(0) = v$). Then $\exp_x tv = \gamma_v(t, x)$ for each real number $t$.

## 2. Definitions and Notations

**Definition 2.1 (Fixed Point).** Let $X$ be a non empty set and $T : X \to X$ be a mapping. The point $x \in X$ is said to be a fixed point of $T$ if $x$ remains invariant under $T$, i.e. $Tx = x$.

**Example 2.2.**

1. A translation mapping has no fixed point i.e. $Tx = x + 1$ for all $x \in R$.
2. The mapping $T : R \to R$ defined by $Tx = \frac{x}{3} - 2$, $x = -3$, is unique fixed point.
3. A mapping $T : R \to R$ defined by $Tx = x^2$ has two fixed points 0 and 1.
4. A mapping $T : R \to R$ defined by $Tx = x$ has infinitely many fixed points i.e. every point of $R$ is a point of $R$.

**Definition 2.3 (Euclidean Space).** Euclidean space is a finite dimensional real vector space $R^n$ with an inner product $(x, y)$, $x, y \in R^n$, which is a suitable chosen (Cartesian) coordinate system

$$x = (x_1, x_2, \ldots, x_n)$$

$$y = (y_1, y_2, \ldots, y_n)$$

is given by the formula $(x, y) = \sum_{i=1}^n x_iy_i$.

**Definition 2.4 (Manifold).** Manifold is a topological space that is locally Euclidean.

**Definition 2.5 (Riemannian Manifold).** Riemannian manifold or Riemannian space $(M, g)$ a real smooth manifold $M$ equipped with an inner product on the tangent space.

**Definition 2.6 (Hadamard Manifold [13]).** Hadamard manifold named after Jacques Hadamard sometimes called a Cartan-Hadamard manifold after Elie carter is a Riemannian manifold $(M, g)$ that is complete and simply connected, and has everywhere non-positive sectional curvature.
Example 2.7.

(1) The real line $\mathbb{R}$ with its usual metric is a Hadamard manifold with constant sectional curvature equal to 0.

(2) Standard $n$-dimensional hyperbolic space $H^n$ is a Hadamard manifold with constant sectional curvature equal to -1.

**Definition 2.8** (Equilibrium Problem [13]). For a given bifunction $F(\cdot, \cdot) : K \times K \to \mathbb{R}$, the problem of finding $u \in K$ such that

$$F(u, v) \geq 0 \quad \forall \quad v \in K, \quad (1)$$

is called equilibrium problem on Hadamard manifolds.

**Definition 2.9** (Firmly non expansive Mapping [24]). Given a mapping $T : K \to K$ defined on $K \subseteq M$, we say that $T$ is firmly non-expansive if for any $x, y \in K$, the function $\phi : [0, 1] \to [0, \infty]$ defined by

$$\phi(t) = d(\gamma_1(t), \gamma_2(t))$$

is non-increasing, where $\gamma_1$ and $\gamma_2$ denote the geodesics joining $x$ to $T(x)$ and $y$ to $T(y)$, respectively. Every firmly non expansive mapping is non expansive, that is, for all $x, y \in K$

$$d(T(x), T(y)) \leq d(x, y).$$

**Definition 2.10** (Fejer Monotone Sequence [24]). Let $X$ be a complete metric space and $C \subseteq X$ be a non empty set. A sequence $\{x_n\} \subset X$ is called Fejer montone w.r.t. $C$ if

$$d(x_{n+1}, y) \leq d(x_n, y) \quad \text{for all } y \in C \text{ and } n \geq 1.$$

**Definition 2.11** (Resolvent of bifunction [24]). Let $F : K \times K \to \mathbb{R}$. For any $\lambda > 0$, the resolvent of $F$ is the set-valued operator $J^F_\lambda : M \to 2^K$ defined by

$$J^F_\lambda(x) = \{z \in K | \lambda F(z, y) - \langle \exp_{x}^{-1}x, \exp_{y}^{-1}y \rangle \geq 0, \forall \ y \in K\}, \quad \forall \ x \in M.$$

**Definition 2.12** (Geodesic Convex Function [26]). A real valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set $K$ is said to be geodesic convex if and only if for $0 \leq t \leq 1$

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

**Definition 2.13** (KKM Mapping [10]). Let $K \subset M$ be a non empty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping. We say that $G$ is KKM mapping if for any $\{x_1, \ldots, x_m\} \subset K$, we have $C_0(\{x_1, x_2, \ldots, x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$

**Definition 2.14** (Hemi continuous Function [20]). A function $F : K \to R$ is said to be hemi-continuous if for every geodesic $\gamma : [0, 1] \to K$, whenever $t \to 0$, $F(\gamma(t)) \to F(\gamma(0))$.

**Definition 2.15** (Fixed Point Property [26]). A topological space $T$ is of the fixed point property if every continuous function $f : T \to T$ has a fixed point.
Definition 2.16 (Variational inequality [26]). For a given single valued vector field $T : M \to TN$. Consider the problem of finding $u \in K$ such that

$$\langle Tu, \exp^{-1} u \rangle \geq 0 \quad \forall \ v \in K,$$

which is called the variational inequality.

Definition 2.17 (Upper semi-continuous [26]). Given $T : M \to 2^M$ and $x_0 \in M$, the mapping $T$ is said to be

1. Upper semi-continuous (USC), at $x_0$ if for any open set $V \subseteq M$ satisfying $T(x_0) \subseteq V$, there exists an open neighbourhood $U(x_0)$ of $x_0$ such that $T(x) \subseteq V$ for any $x \in U(x_0)$.

2. Upper Kuratowski semicontinuous (UKSC), at $x_0$ if for any sequences $\{x_k\}, \{u_k\} \subset M$ with each $u_k \in T(x_k)$, the relation $\lim_{k \to \infty} x_k = x_0$ and $\lim_{k \to \infty} u_k = u_0$ imply $u_0 \in T(x_0)$.

3. Equilibrium Problems on Hadamard Manifolds

In this section, we present some algorithm for equilibrium problems on Hadamard manifolds proved by Vittorio Colao et al. [24], M.A. Noor et al. [13] using the auxiliary principle technique.

3.1. Existence of Equilibrium Points

An equilibrium theory in Euclidean spaces was first introduced by Ky Fan in [8, 9] and then developed by Brezis, Nirenboag and Stampacchia [6] among others. In order to get an existence result for this equilibrium problem they provide following analogues to KKM Lemma [1] in the setting of Hadamard manifolds.

Lemma 3.1. Let $G : K \to 2^K$ be a mapping such that or each $x \in K$, $G(x)$ is closed suppose that

1. there exist $x_0 \in K$ such that $G(x_0)$ is compact

2. $\forall \ x_1, x_2, \ldots, x_m \in K, \ C_0(\{x_1, \ldots, x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$.

Then $\bigcap_{x \in K} F(x) \neq \emptyset$.

Theorem 3.2. Let $F : K \times K \to R$ be a bifunction such that

1. for any $x \in K$, $F(x, x) \geq 0$;

2. for every $x \in K$, the set $\{y \in K : F(x, y) < 0\}$ is convex

3. for every $y \in K$, $x \to F(x, y)$ is upper semicontinuous.

4. there exist a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that $F(x, y_0) < 0, \forall \ x \in K/L$.

Then there exist a point $x_0 \in L \cap K$ satisfying $F(x_0, y) \geq 0, \forall \ y \in K$.

By setting $L = K$ in the previous theorem, the following corollary is obtained.

Corollary 3.3. Let $K \subseteq M$ be convex and compact and $F : K \times K \to R$ such that

1. for any $x \in K$, $F(x, x) \geq 0$;

2. for every $x \in K$ the set $\{y \in K : F(x, y) < 0\}$ is convex
(3). for every $y \in K$, $x \to F(x, y)$ is upper semicontinuous.

Then there exist a point $x_0 \in K$ satisfying $F(x_0, y) \geq 0$, $\forall \ y \in K$.

**Example 3.4.** Example of an equilibrium problem defined in a Euclidean space whose set $K$ is not convex so it cannot be solved by using the classical results known in vector spaces. However, if we rewrite the problem in a Riemannian manifold then it turns out to satisfy the conditions required in the Corollary 3.3. Let

$$K = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, y^2 - z^2 = -1, z \geq 0\}$$

and $F : K \times K \to R$ the bifunction defined by

$$F(x_1, y_1, z_1, x_2, y_2, z_2) = 4(x_2 - x_1) + (1 - x_1)((y_2^2 + z_2^2) - (y_1^2 + z_1^2))$$

Note that $K$ is indeed not convex in $\mathbb{R}^3$. Given a natural number $m \geq 1$. Let $E^{m,1}$ denote the vector space $\mathbb{R}^{m+1}$ endowed with the symmetric bilinear form (which is called the Lorentz metric) defined by

$$(x, y) = \sum_{i=1}^{m} x_i y_i - x_{m+1} y_{m+1}, \forall \ x = (x_i), y = (y_i) \in \mathbb{R}^{m+1}$$

The hyperbolic $m$-space $H^m$ is defined by

$$\{x = (x_1, \ldots, x_{m+1}) \in E^{m,1} : \langle x, x \rangle = -1, x_{m+1} > 0\},$$

that is upper sheet of the hyperboloid $\{x \in E^{m,1} : \langle x, x \rangle = -1\}$. Note that $x_{m+1} \geq 1$ for any $x \in H^m$, with equality if and only if $x_i = 0$ for all $i = 1, \ldots, m$. the metric of $H^m$ is induced from the Lorentz metric $(\cdot, \cdot)$ and it will be denoted by the same symbol. Then $H^m$ is a Hadmard manifold sectional curvature $-1$ (c.f. [15]). Furthermore, the normalized geodesic $\gamma : R \to H^m$ starting from $x \in H^m$ is given by $\gamma(t) = (\cosh(t))x + (\sinh(t))v$, $\forall \ t \in R$, where $v \in T_x H^m$ is a unit vector.

Considering the set $K$ immersed in the space $M = \mathbb{R} \times H^1$ which is a Hadmard manifold for being the product space of Hadamard manifolds (cf. [15]), it is readily seen that $K$ is convex and compact on $M$. On the other hand, conditions (i) and (iii) in Corollary 3.3 hold, and the fact that $F$ is convex is the second variable. So Corollary 3.3 implies the existence of an equilibrium point for $F$.

**Theorem 3.5.** Let $K \to TM$ be a continuous vector field and $f : K \to R$ a convex lower semicontinuous function. Assume that the following condition holds: (C) There exists a compact set $L \subseteq M$ and a point $y_0 \in L \cap K$ such that

$$(Ax, \exp^{-1}_x y_0) + f(y_0) - f(x) < 0, \forall \ x \in K \setminus L.$$

Then MVIP$(A, f)$ has a solution in $L \cap K$.

**Corollary 3.6.** Let $A : K \to TM$ be a continuous vector field and $f : K \to R$ a convex lower semicontinuous function. If either

(i). $K$ is compact, or

(ii). there exists $y_0 \in K$ such that the coercivity condition

$$\frac{\langle Ay_0, \exp^{-1}_{y_0} x \rangle + \langle Ax, \exp^{-1}_{y_0} y_0 \rangle}{d(y_0, x)} \to -\infty \text{ as } d(y_0, x) \to \infty$$

holds, then MVIP $(A, f)$ has a solution.
Remark 3.7. By considering $f$ the function constantly 0, it follows that Corollary 3.6 extended.

Lemma 3.8. Let $D, K \subseteq M$ be closed convex sets with $D$ compact. Assume that $\rho : D \times K \to R$ is upper semicontinuous in the first variable and that for any $x \in D$ and $y \in K$, $-\rho(x, \gamma) + \rho(x, \cdot)$ are convex functions. If $\max_{x \in D} \rho(x, y) \geq 0$, $\forall$ $y \in K$ then there exists $\hat{x} \in D$ such that $\rho(\hat{x}, y)$ for any $y \in K$.

Theorem 3.9. Let $K \subseteq M$ be a compact convex set and $T : K \to 2^K$ and UKSC mapping. Assume that for any $x \in K$, $T(x)$ is closed and convex. Then there exists a fixed point of $T$.

Remark 3.10. The upper semicontinuity implies the upper Kuratowski semicontinuity, so the previous result remains true assume that $T$ is USC instead.

3.2. Approximation of Equilibrium Points

The approach that followed to approximate a solution of the equilibrium problem (for the bifunction $F$ and the set $K$ find $x \in K$ such that $F(x, y) \geq 0$, $\forall y \in K$) involves the resolvent of the bifunction $F$, which is firmly non-expansive mapping whose fixed point set coincides with the equilibrium point set of $F$.

Proposition 3.11 ([2]). A mapping $T : K \to K$ is firmly nonexpansive iff for any $x, y \in K$

$$\left\langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x \right\rangle + \left\langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y \right\rangle \leq 0$$

As in Banach spaces and the Hilbert Ball [20], the class of firmly nonexpansive mappings is characterized by the good asymptotic behaviour of the sequence of Picard iterates $\{T^n(x)\}$. In order to prove the convergence of this sequence, the following definition and results are necessary.

Lemma 3.12 ([4, 17]). Let $X$ be a complete metric space. If $\{x_n\} \subset X$ is Fejér monotone with respect to a non empty set $C \subseteq X$, then $\{x_n\}$ is bounded. Moreover, if a cluster $x$ of $\{x_n\}$ belong to $C$, then $\{x_n\}$ converges to $x$.

Theorem 3.13. Let $T : K \to K$ be a firmly nonexpansive mapping such that its fixed point set $\text{Fix}(T) \neq \emptyset$. Then for each $x \in k$ the sequence of iterates $\{T^n(x)\}$ converges to a fixed point of $T$.

3.3. Resolvents of Bifunction

The definition of the resolvent of a bifunction in the setting of a Hilbert space $H$ appears implicitly in [3] and was first given in [18]. In order to distinguish the resolvent of vector fields and the resolvent of bifunctions, denoted latter with an upper index, $J^F$. Given a bifunction $F : K \times K \to R$, where $K \subseteq H$ is non-empty closed and convex, the resolvent of $F$ is the set-valued operator $J^F : H \to 2^K$ such that for any $x \in H$, $J^F(x) = \{z \in K | (\forall y \in K) F(z, y) + (z - x, y - z) \geq 0\}$. Under some conditions on the bifunction $F$, $J^F$ can be proved to be well defined, single-valued and firmly nonexpansive, and its fixed point set turns out to be the equilibrium point set of $F$.

The following definition extends the previous one to the setting of a Hadamard manifold $M$.

Definition 3.14. Let $F : K \times K \to R$. For any $\lambda > 0$, the resolvent of $F$ is the set-valued operator $J^F_\lambda : M \to 2^K$ defined by

$$J^F_\lambda(x) = \{z \in K | \lambda F(z, y) - \langle \exp_{x}^{-1} x, \exp_{y}^{-1} y \rangle \geq 0, \forall y \in K\}, \forall x \in M.$$

Theorem 3.15. Let $F : K \times K \to R$ be a bifunction satisfying the following conditions:

(1). $F$ is monotone, that is, for any $(x, y) \in K \times K$, $F(x, y) + F(y, x) \leq 0$;
Since in terms of the distance and the exponential map, the above inequality can be rewritten as
\[ \gamma_{xy} \in x, y \] for any \( \lambda > 0 \).

Remark 3.16. The resolvent could be defined for a set-valued bifunction \( F : K \times K \rightarrow 2^K \) as the set-valued function \( J^F_{\lambda} : M \rightarrow 2^K \) such that
\[ J^F_{\lambda}(x) = \{ z \in K | \lambda u - \langle \exp^{-1} x, \exp^{-1} y \rangle \geq 0, \forall y \in K, \forall u \in F(z, y) \}, \]
for any \( \lambda > 0 \) and any \( x \in M \). Then, assuming that \( F \) monotone means that \( u + v \leq 0 \) for any \( u \in F(x, y) \), \( v \in F(y, x) \) and \( x, y \in K \), the previous theorem would remain true except for (iii) which needs \( F \) to be single-valued.

3.4. Auxiliary Principle Technique

Muhammad Aslam Noor and Khalida Inayat Noor [13] suggested and analyzed an iterative method for solving the equilibrium problems on Hadamard manifolds using the auxiliary principle technique. They considered the convergence analysis of this condition.

Definition 3.17 (Tangent Space). Let \( M \) be a simply connected \( m \)-dimensional manifold. Given \( x \in M \), the tangent space of \( M \) at \( x \) is denoted by \( T_x M \) and can be defined as. In differential geometry, one can attach to every point \( x \) of a differentiable manifold a tangent space, a real vector space that intuitively contains the possible “directions” at which one can tangentially pass through \( x \). The elements of the tangent space are called tangent vectors at \( x \).

Lemma 3.18. Let \( x \in M \). Then \( \exp : T_x M \rightarrow M \) is a diffeomorphism, and for any two points \( x, y \in M \), there exist a unique normalized geodesic joining \( x \) to \( y, \gamma_{xy} \), which is minimal.

Lemma 3.19. Comparison theorem for triangles. Let \( \Delta(x_1, x_2, x_3) \) be a geodesic triangle. Denote for each \( i = 1, 2, 3 \) (mod 3), by \( \gamma_i : [0, l_i] \rightarrow M \) the geodesic joining \( x_i \) to \( x_{i+1} \) and \( \alpha_i = L(\gamma_i'(0) - \gamma_i'(i-1)(l_i, 1)) \), the angle between the vectors \( \gamma_i'(0) \) and \( -\gamma_i'(-1)(l_i, -1) \) and \( l_i = L(\gamma_i) \). Then
\[ \alpha_1 + \alpha_2 + \alpha_3 \leq \pi, \quad l_1^2 + l_2^2 + l_3^2 - 2L_1L_2 \cos \alpha_2 \leq l_3^2. \]
The above inequality can be rewritten as
\[ d^2(x_1, x_2) + d^2(x_2, x_3) - 2(\exp_{x_{i+1}}^{-1} \exp^{-1}_{x_{i+1}, x_{i+2}} x_{i+2}) \leq d^2(x_{i-1}, x). \]

Since
\[ \langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \rangle = d(x_i, x_{i+1})d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}, \]
Lemma 3.20. Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold $M$. Then there exists $x', y', z' \in \mathbb{R}^2$ such that

\[
d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|
\]

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of $M$.

Lemma 3.21. Let $M$ be a Hadamard manifold and $f : M \to \mathbb{R}$ be convex. Then for any $x \in M$, the subdifferential $\partial f(x)$ is non-empty. That is, $D(\partial f) = M$.

3.5. Implicit Iterative Method

Noor et al. [14] used the auxiliary principle technique of Glowinski et al. [19] to analyze an implicit iterative method for solving the equilibrium problem. For given $u \in K$ satisfying equilibrium problem (1). Consider the problem of finding $w \in K$ such that

\[
\rho F(u, v) + \langle \exp^{-1} u, \exp^{-1} v \rangle \geq 0, \quad \forall \ v \in K
\]

which is called the auxiliary problem on Hadamard manifolds.

Algorithm 3.22. For a given $u_0$, compute the approximate solution by the iterative scheme

\[
\rho F(u_n, v) + \langle \exp^{-1} u_{n+1}, \exp^{-1} v \rangle \geq 0, \quad \forall \ v \in K
\]

is called the explicit iterative method for solving the equilibrium problem on the Hadamard manifold.

Algorithm 3.23. For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme

\[
\langle \rho T u_n + \exp^{-1} u_{n+1}, \exp^{-1} v \rangle \geq 0, \quad \forall \ v \in K
\]

For $M = \mathbb{R}^n$, Algorithm 3.23 reduces to

Algorithm 3.24. For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme

\[
\langle \rho T u_n + u_{n+1} - u_n v - u_n, v \rangle \geq 0, \quad \forall \ v \in K
\]

Theorem 3.25. Let $F(\cdot, \cdot)$ be a partially relaxed strongly monotone bifunction with a constant $\alpha > 0$. Let $u_n$ be the approximate solution of the equilibrium problem (1) obtained from Algorithm 3.22, then

\[
d^2(u_{n+1}, u) \leq d^2(u_n, u) - (1 - \rho \alpha) d^2(u_n, u),
\]

where $u \in K$ is a solution of the equilibrium problem.

Glowinski et al. [19] suggested and analyzed an implicit iterative method for solving the equilibrium problem (1). For a given $u \in K$ satisfying (1), consider the problem of find $w \in K$ such that

\[
\rho F(w, v) + \langle \exp^{-1} u, \exp^{-1} v \rangle \geq 0, \quad \forall \ v \in K
\]

which is called the auxiliary equilibrium problem on Hadamard manifolds. They have shown that the convergence analysis of this method requires only the pseudomonotonicity which is a weaker condition than monotonicity.
Algorithm 3.26. For a given $u_0$, compute the approximate solution by the iterative scheme

$$\rho F(u_{n+1}, v) + \left\langle \exp^{-1}_{u_n} u_{n+1}, \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \quad \forall \ v \in K$$

is called the implicit (proximal point) iterative method for solving the equilibrium problem on the Hadamard manifold.

Algorithm 3.26 can be written in the following equivalent form.

Algorithm 3.27. For a given $u_0 \in K$, find the approximate solution $u_{n+1}$ by the iterative scheme

$$\rho F(u_n, v) + \left\langle \exp^{-1}_{u_n} y_n, \exp^{-1}_{y_n} v \right\rangle \geq 0, \quad \forall \ v \in K$$

$$\rho F(y_n, v) + \left\langle \exp^{-1}_{y_n} \exp^{-1}_{u_{n+1}} v \right\rangle \geq 0, \quad \forall \ v \in K$$

is a two-step iterative method for solving the equilibrium problem on Hadamard manifolds. This method can be viewed as the extragradient method for solving the equilibrium problems.

If $K$ is a convex set in $\mathbb{R}^n$, then Algorithm 3.26 collapses to the following

Algorithm 3.28. For a given $u_0 \in K$, find the approximate solution $u_{n+1}$ by the iterative scheme:

$$\rho F(u_{n+1}, u) + (u_{n+1} - u_n, v - u_{n+1}) \geq 0, \quad \forall \ v \in K$$

which is known as the implicit method for solving the equilibrium problem.

For the convergence analysis of Algorithm 3.27, see [11, 12]. If $F(u, v) = (Tu, \exp^{-1}_{u_n} v)$, where $T$ is a single valued vector field $T : K \to TM$, then Algorithm 3.26 reduces to the following implicit method for solving the variational inequalities.

Algorithm 3.29. For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$\langle \rho Tu_n + \exp^{-1}_{u_n} y_n, \exp^{-1}_{y_n} u_{n+1} \rangle \geq 0, \quad \forall \ v \in K$$

Algorithm 3.29 is due according to Tang et al. [5] and M. A. Noor and K. I. Noor [13]. We can also rewrite Algorithm 3.29 in the following equivalent form.

Algorithm 3.30. For a given $u_0 \in K$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$\langle \rho Tu_n + \exp^{-1}_{u_n} y_n, \exp^{-1}_{y_n} u_{n+1} \rangle \geq 0, \quad \forall \ v \in K$$

$$\langle \rho Ty_n + \exp^{-1}_{y_n} u_n, \exp^{-1}_{u_{n+1}} v \rangle \geq 0, \quad \forall \ v \in K$$

which is the extragradient method for solving the variational inequalities on Hadamard manifolds and appears to be a new one.

In a similar way, one can obtain several iterative methods for solving the variational inequalities on the Hadamard manifold. We now consider the convergence analysis of Algorithm 3.26 and this is the motivation of this next result.

Theorem 3.31. Let $F(\cdot, \cdot)$ be a pseudomonotone bifunction. Let $u_n$ be the approximate solution of the equilibrium problem obtained from Algorithm 3.26, then

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) \leq d^2(u_n, u)$$

where $u \in K$ is a solution of the equilibrium problem (1).
Theorem 3.32. Let \( u \in K \) be solution of (1) and let \( u_{n+1} \) be the approximate solution obtained from Algorithm 3.26, then
\[
\lim_{n \to \infty} u_{n+1} = u.
\]

In the next section, we study the existence of solutions of mixed equilibrium problems on Hadamard manifolds. S. Jena et al. [20] introduced the implicit and explicit algorithms to solve these problems. They showed that the sequence generated by both implicit and explicit algorithms converges to a solution of mixed equilibrium problems, whenever it exists, under reasonable assumptions.

3.6. Existence of Solutions of Mixed Equilibrium Problems

Colao et al. [24] studied existence of solutions of equilibrium problems under monotonicity assumptions on Hadamard manifolds.

Definition 3.33 (Mixed Equilibrium Problem). Let \( \psi : K \to \mathbb{R} \) be a mapping and \( F : K \times K \to \mathbb{R} \) be a bifunction satisfying the property \( F(x,x) = 0 \) for all \( x \in K \). Then the problem is to find \( \bar{x} \in K \) such that
\[
F(\bar{x}, y) + \psi(y) - \psi(x) \geq 0 \quad \forall \ y \in K
\]  

is called mixed equilibrium problem. Colao et al. [4] called a bifunction \( F \) to be monotone on \( K \) if for any \( x, y \in K \), we have \( F(x, y) + F(y, x) \leq 0 \). A bifunction \( F \) is said to be pseudomonotone with respect to the function \( \psi \) if
\[
F(x, y) + \psi(y) - \psi(x) \geq 0
\]
\[
\Rightarrow F(y, x) + \psi(x) - \psi(y) \leq 0
\]

Lemma 3.34. Let \( F : K \times K \to \mathbb{R} \) be hemicontinuous in the first argument and for fixed \( x \in K \) the mapping \( z \to F(x, z) \) be geodesic convex. Also assume that the map \( \psi : K \to \mathbb{R} \) is geodesic convex and the bifunction \( F \) is pseudomonotone with respect to \( \psi \). Then \( \bar{x} \in K \) is a solution of the mixed equilibrium problem (2) if and only if \( F(y, \bar{x}) + \psi(\bar{x}) - \psi(y) \leq 0 \) for all \( y \in K \).

Theorem 3.35. Let \( K \) be a bounded subset of \( M \) and \( F : K \times K \to \mathbb{R} \) be hemicontinuous in the first argument. Suppose for fixed \( x \in K \), the mapping \( z \to F(x, z) \) and \( \psi : K \to \mathbb{R} \) are geodesic convex, lower semicontinuous. Also assume that the bifunction \( F \) is pseudomonotone with respect to \( \psi \). Then the mixed equilibrium problem (2) has a solution.

Theorem 3.36. Let \( K \) be an unbounded subset of \( M \) and \( F : K \times K \to \mathbb{R} \) be hemicontinuous in the first argument. Suppose for fixed \( x \in K \), the mapping \( z \to F(x, z) \) and \( \psi : K \to \mathbb{R} \) are geodesic convex, lower semicontinuous. Also assume that the bifunction \( F \) is pseudomonotone with respect to \( \psi \). If there exists a point \( x_0 \in K \), such that \( F(x, x_0) + \psi(x_0) - \psi(x) < 0 \), whenever \( d(0, x) \to +\infty \), \( x \in K \) holds, then the mixed equilibrium problem (2) has a solution.

Now we present some Implicit methods for solving mixed equilibrium problem:

Algorithm 3.37. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \) as a solution of the following iterative scheme.
\[
F(x_{n+1}, y) + \frac{1}{\rho} \left\langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \right\rangle + \psi(y) - \psi(x_{n+1}) \geq 0 \quad \forall \ y \in K
\]  

(i). When \( \psi \equiv 0 \), the Algorithm 3.37 reduces to the following implicit iterative algorithm.
Algorithm 3.38. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), such that

\[
F(x_{n+1}, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0 \quad \forall \ y \in K
\]

This is the implicit algorithm for the equilibrium problems introduced by Noor et al. [11].

(ii). If \( K \) is convex set in \( \mathbb{R}^n \), then Algorithm 3.37 reduces into the following algorithm [11, 12].

Algorithm 3.39. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), as a solution of the iterative scheme

\[
F(x_{n+1}, y) + \frac{1}{\rho} (x_{n+1} - x_n, y - x_{n+1}) + \psi(y) - \psi(n+1) \geq 0 \quad \forall \ y \in K
\]

(iii). If we take \( F(x, y) = \langle V_x, \exp_{x_n}^{-1} y \rangle \), then Algorithm 3.37 reduces to the following.

Algorithm 3.40. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), as a solution of the iterative scheme

\[
\langle \rho V x_{n+1} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n+1}) \geq 0 \quad \forall \ y \in K,
\]

which is an algorithm for solving mixed variational inequalities studied by Noor et al. [13].

(iv). When \( \psi \equiv 0 \), the Algorithm 3.40 reduces to the following implicit iterative algorithm for solving variational inequalities.

Algorithm 3.41. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), as a solution of the iterative scheme

\[
\langle \rho V x_{n+1} + \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0 \quad \forall \ y \in K
\]

Theorem 3.42. Let \( F : K \times K \to \mathbb{R} \) be pseudomonotone with respect to the function \( \psi \) and continuous in the first argument and SOL(MEP) \( \neq \phi \). Suppose that the sequence \( \{x_n\} \) generated by (3) is well defined and \( \psi : K \to \mathbb{R} \) is continuous. Then \( \{x_n\} \) converges to a solution of the mixed equilibrium problem (2).

Some Explicit methods for solving mixed equilibrium problem:

Definition 3.43. The bifunction \( F \) is said to be partially relaxed pseudomonotone with respect to the function \( \psi \) if there exists \( \alpha > 0 \) such that \( \forall \ x, y, z \in K \),

\[
F(x, y) + \psi(y) - \psi(z) \geq 0 \Rightarrow F(z, x) + \psi(x) - \psi(z) \leq \alpha d^2(y, z)
\]

If we take \( z = y \), then \( F \) reduces to a pseudomonotone function.

Algorithm 3.44. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), as a solution of the iterative scheme

\[
F(x_n, y) + \frac{1}{\rho} \langle \exp_{x_n}^{-1} x_{n+1}, \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_n) \geq 0 \quad \forall \ y \in K
\]

Some particular cases of Algorithm 3.44 are given as follows:

(i). When \( \psi \equiv 0 \), the Algorithm 3.44 reduces to the following explicit iterative algorithm for equilibrium problems.

Algorithm 3.45. At stage \( n \), given \( x_n \in K \), \( \rho > 0 \), compute \( x_{n+1} \in K \), such that

\[
F(x_n, y) + \frac{1}{\rho} \langle \exp_{x_{n+1}}^{-1}, \exp_{x_{n+1}}^{-1} y \rangle \geq 0 \quad \forall \ y \in K.
\]
Algorithm 3.46. At stage $n$, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$F(x_n, y) + \frac{1}{\rho} < x_{n+1} - x_n, y - x_n > + \psi(y) - \psi(x_{n+1}) \geq 0 \ \forall \ y \in K.$$ 

Algorithm 3.47. At stage $n$, given $x_n \in K$, $\rho > 0$ compute $x_{n+1} \in K$ as a solution of the iterative scheme

$$\langle \rho Vx_n + \exp_{x_n}^{-1} \exp_{x_{n+1}}^{-1} y \rangle + \psi(y) - \psi(x_{n+1}) \geq 0, \ \forall \ y \in K.$$ 

Algorithm 3.48. At stage $n$, given $x_n \in K$, $\rho > 0$, compute $x_{n+1} \in K$, as a solution of the iterative scheme

$$\langle \rho Vx_n + \exp_{x_n}^{-1} \exp_{x_{n+1}}^{-1} y \rangle \geq 0, \ \forall \ y \in K.$$ 

Algorithm 3.49. Let $F : K \times K \to R$ be a partially relaxed pseudomonotone bifunction with respect to the function $\psi$ with a constant $\alpha > 0$, and continuous in the first argument. Suppose that the sequence $\{x_n\}$ generated by (4) is well defined, $\psi : K \to R$ is continuous and $SOL(MEP) \neq \emptyset$. Then

$$d^2(x_{n+1}, x) \leq d^2(x_n, x) - (1 - 2\rho\alpha)d^2(x_{n+1}, x_n)$$

If in addition $\rho < \frac{1}{2\alpha}$, then $\{x_n\}$ converges to a solution of the mixed equilibrium problem (2).

References

On the Geometry of Pseudo-slant Submanifolds of LP-Cosymplectic Manifold

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Abstract: In this paper, we study pseudo-slant submanifolds of LP-cosymplectic manifold and obtained some interesting characterization results.

MSC: 53C42, 53C25, 53C40.

Keywords: Slant submanifold, pseudo-slant submanifold and LP-cosymplectic manifold.

1. Introduction

The slant immersion was initially studied by Chen ([6], [7]) as a generalization of both holomorphic and totally real submanifolds in almost Hermitian manifolds. Lotta [15] extended the concept of slant immersion into almost contact metric manifolds. As generalization, Papaghiuc [17](resp. Cabrerizo et al. [5]) introduced a new class of submanifolds called semi-slant submanifolds which is the generalization of CR-submanifolds and slant submanifolds in almost Hermitian manifold(resp. almost contact metric manifolds). Carriazo introduced and studied bi-slant submanifolds in almost Hermitian manifolds and simultaneously to anti-slant submanifolds in almost Hermitian manifolds [4]. Later V.A. Khan et al. renamed it as pseudo-slant submanifolds [11]. Since then many geometers like ([8], [11], [12], [13], [19], [20]) have studied slant and their generalized submanifolds of various manifolds. Motivated by these studies we study pseudo-slant submanifolds of LP-cosymplectic manifolds.

The paper is organized as follows; In section 2, we give a brief account of LP-cosymplectic manifold and submanifold. In section 3, we consider $M$ is a pseudo-slant submanifold of LP-cosymplectic manifold $\tilde{M}$ and proved some characterization results.

2. Preliminaries

Let $\tilde{M}$ be an a $(2n+1)$-dimensional paracontact manifold, equipped with a Lorentzian paracontact metric structure $(\phi, \xi, \eta, g)$ that is $\phi$ a (1,1) tensor field, $\xi$ a vector field, $\eta$ a contact form and $g$ Lorentzian metric of type (0,2) with signature

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\[ \phi^2(X) = X + \eta(X)\xi, \quad (1) \]
\[ \phi\xi = 0, \quad \eta \cdot \phi = 0, \quad \eta(\xi) = -1, \quad (2) \]
\[ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad (3) \]

\( \forall X, Y \in \Gamma(TM) \) \[16\]. A Lorentzian paracontact metric structure on \( \tilde{M} \) is said to be Lorentzian cosymplectic as simply LP-cosymplectic manifold, if

\[ (\tilde{\nabla}_X \phi)Y = 0, \quad (4) \]

for any vector fields \( X, Y \) on \( \tilde{M} \), where \( \tilde{\nabla} \) denotes the Levi-Civita connection with respect to \( g \). From (1) and (4), it follows that

\[ (\tilde{\nabla}_X \xi) = 0, \quad (5) \]

for all \( X, Y \in \Gamma(T\tilde{M}) \). Now, let \( M \) be a submanifold of a Lorentzian almost paracontact manifold \( \tilde{M} \) with the induced metric \( g \). Then the Gauss and Weingarten formulae are given by

\[ \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (6) \]
\[ \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (7) \]

respectively, where \( \nabla \) and \( \nabla^\perp \) are the induced connections on the tangent bundle \( TM \) and normal bundle \( T^\perp M \) of \( M \) respectively. Here \( \sigma \) and \( A_V \) are the second fundamental form and shape operator corresponding to the normal vector field \( V \) and are related by

\[ g(A_V X, Y) = g(\sigma(X, Y), V) \quad (8) \]

for all \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \). A submanifold \( M \) of a Lorentzian almost paracontact manifold \( \tilde{M} \) is said to be totally umbilical if

\[ \sigma(X, Y) = g(X, Y)H, \quad (9) \]

where \( H \) is the mean curvature.

3. Pseudo-slant Submanifold of a LP-cosymplectic Manifold

In this section, we consider \( M \) is a pseudo-slant submanifold of a LP-cosymplectic manifold and obtain some characterization results.

For \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \), we can set

\[ \phi X = TX + FX \text{ and } \phi V = tV + fV, \quad (10) \]

where \( TX \) (respectively \( tV \)) and \( FX \) (respectively \( fV \)) are the tangential and normal components of \( \phi X \) (respectively \( \phi V \)). A submanifold \( M \) is said to be invariant (respectively anti-invariant) if \( F \) (respectively \( T \)) is identically zero i.e., \( \phi X \in \Gamma(TM) \) (respectively \( \phi X \in \Gamma(T^\perp M) \)) for all \( X \in \Gamma(TM) \). Thus by using (1) and (10) one can get

\[ T^2 = I - tF + \eta \otimes \xi \quad FT + fF = 0, \quad (11) \]
\[ f^2 = I - fT \quad Tt + tf = 0. \quad (12) \]
We define the covariant derivatives of the tensor field \( T, F, t \) and \( f \) as

\[
(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{13}
\]
\[
(\nabla_X F)Y = \nabla_X FY - F\nabla_X Y, \tag{14}
\]
\[
(\nabla_X t)V = \nabla_X tV - t\nabla_X V, \tag{15}
\]
\[
(\nabla_X f)V = \nabla_X fV - f\nabla_X V, \tag{16}
\]
respectively. Moreover, for any \( X,Y \in \Gamma(TM) \), we have \( g(TX,Y) = g(X,TY) \) and for \( U,V \in \Gamma(T^\perp M) \), we have \( g(U,fV) = g(fU,V) \). These imply that \( T \) and \( f \) are also symmetric tensor fields. Further for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \), we have

\[
g(FX,V) = g(X,tV), \tag{17}
\]
which is the relation between \( F \) and \( t \). The Gauss and weingarten formulas together with (4) and (10) yield

\[
(\nabla_X T)Y = A_{FY}X + t\sigma(X,Y), \tag{18}
\]
\[
(\nabla_X F)Y = f\sigma(X,Y) - \sigma(X,TY), \tag{19}
\]
\[
(\nabla_X t)V = A_{fV}X - TA_VX \tag{20}
\]
and

\[
(\nabla_X f)V = -\sigma(tV,X) - FA_VX, \tag{21}
\]
for all \( X,Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \). Since \( \xi \in TM \) by virtue of (5), (6), (8) and (10), we obtain

\[
\nabla_X \xi = 0, \quad \sigma(X,\xi) = 0, \quad A_V\xi = 0, \tag{22}
\]
for all \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

**Definition 3.1** ([7]). Let \( M \) be any submanifold, then \( M \) is said to be slant submanifold if for any \( x \in M \) there exists a constant angle \( \theta \in [0,\pi/2] \) between \( T_xM \) and \( \phi X \) for all \( X \neq 0 \) called slant angle of \( M \). Hence one can see that slant submanifolds are generalization of invariant and anti-invariant submanifolds i.e., if \( \theta = 0 \) then \( M \) becomes invariant and if \( \theta = \pi/2 \) then \( M \) becomes anti-invariant submanifold.

**Theorem 3.2** ([21]). Let \( M \) be a submanifold of an LP-cosymplectic manifold \( \tilde{M} \) such that \( \xi \) is tangent to \( M \). Then \( M \) is slant submanifold if and only if there exists a constant \( \lambda \in [0,1] \) such that

\[
T^2 = \lambda(I + \eta \bigotimes \xi). \tag{23}
\]

Furthermore, in this case if \( \theta \) is the slant angle of \( M \), then \( \lambda = \cos^2\theta \).

From [5], we have

\[
g(TX,TY) = \cos^2\theta g(X,Y) + \eta(X)\eta(Y), \tag{24}
\]
\[
g(FX,FY) = \sin^2\theta g(X,Y) + \eta(X)\eta(Y) \tag{25}
\]
Let $M$ be a slant submanifold of a lorentzian paracontact manifold $\tilde{M}$ with slant angle $\theta$. Then for any $X \in \Gamma(TM)$, from (11) and (23), we have

$$tFX = \sin^2 \theta (X + \eta(X)\xi)$$  \hspace{1cm} (26)

and from (25), we have

$$F^2 X = \sin^2 \theta (X + \eta(X)\xi).$$  \hspace{1cm} (27)

From (26) and (27), we obtain $F^2 = tF$.

**Definition 3.3 ([11]).** A submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$ is said to be pseudo-slant submanifold if there exists two orthogonal distributions $D^\theta$ and $D^\perp$ on $M$ such that

1. $TM = D^\theta \oplus D^\perp \oplus <\xi>$,
2. The distribution $D^\theta$ is slant distribution with slant angle $\theta \in [0, \pi/2]$,
3. The distribution $D^\perp$ is anti-invariant i.e., $\phi X \in T^\perp M, \forall X \in TM$.

**Theorem 3.4.** Let $M$ be a proper pseudo-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. Then the tensor $F$ is parallel if and only if the tensor $t$ is parallel

**Proof.** By virtue of (8), (18) and (19) we have

$$g((\nabla_X F)Y, V) = g(f\sigma(X, Y), V) - g(\sigma(X, TY), V)$$

$$= g(\sigma(X, Y), fV) - g(A_V X, TY)$$

$$= g(A_{fV} X, Y) - g(A_V X, TY)$$

$$= g(A_{fV} X, Y) - g(TA_V X, Y)$$

$$= g(A_{fV} Y - TA_V X, Y)$$

$$= g((\nabla_X t)V, Y),$$

for every $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. This completes the proof.

**Theorem 3.5.** Let $M$ be a proper pseudo-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. Then the tensor $F$ is parallel if and only if $A_{fV} Y = A_V TY$, for any $Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

**Proof.** For $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, by using (8) and (18) we obtain

$$g((\nabla_X F)Y, V) = g(f\sigma(X, Y), V) - g(\sigma(X, TY), V)$$

$$= g(\sigma(X, Y), fV) - g(A_V TY, X)$$

$$= g(A_{fV} X, Y) - g(A_V TY, X)$$

$$= g(A_{fV} Y, X) - g(A_V TY, X).$$

This completes the proof.

**Theorem 3.6.** Let $M$ be a proper pseudo-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. The covariant derivation of $T$ is symmetric i.e., $g((\nabla_X T)Y, Z) = g((\nabla_X T)Z, Y)$, $\forall X, Y, Z \in \Gamma(TM)$. 

Proof. By using (8), (17) and (18), we have

\[ g((\nabla_X T)Y, Z) = g(AFX + t_\sigma(X, Y), Z) \]
\[ = g(\sigma(X, Z), FY) + g(\sigma(X, Y), FZ) \]
\[ = g(\sigma(X, Z), Y) + g(AFZ, Y) \]
\[ = g(AFZ, X) + g(\sigma(X, Y), Y) \]
\[ = g((\nabla_X T)Z, Y), \]

for any \( X, Y, Z \in \Gamma(TM) \). This proves our assertion.

**Theorem 3.7.** Let \( M \) be a proper pseudo-slant submanifold of a LP-cosymplectic manifold \( \tilde{M} \). Then the tensor \( T \) is parallel if and only if \( AFX = -AFY, X, Y \in \Gamma(TM) \).

**Proof.** By virtue of (8), (17) and (18), we have

\[ g((\nabla_X F)Y, Z) = g(AFX + t_\sigma(X, Y), Z) \]
\[ = g(\sigma(X, Z), FY) + g(\sigma(X, Y), FZ) \]
\[ = g(\sigma(X, Z), X) + g(AFZ, X), \]

for any \( X, Y, Z \in \Gamma(TM) \). This completes the proof.

**Theorem 3.8.** Let \( M \) be a proper pseudo-slant submanifold of a LP-cosymplectic manifold \( \tilde{M} \). The covariant derivation of \( f \) is symmetric i.e., \( g((\nabla_X f)V, U) = g((\nabla_X f)U, V) \), for every \( X \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \).

**Proof.** By virtue of (8), (17) and (21) we have

\[ g((\nabla_X f)V, U) = g(-_\sigma tV, X) - FAV, X, U) \]
\[ = g(-_AV, X, tV) - g(AX, tU) \]
\[ = -g(FAV, X, V) - g(AX, tU, V) \]
\[ = g(-FAV - \sigma(X, tU), V), \]
\[ = g((\nabla_X f)U, V), \]

for any \( X \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \). This proves our assertion.

**Theorem 3.9.** Let \( M \) be a proper pseudo-slant submanifold of a LP-cosymplectic manifold \( \tilde{M} \). Then the tensor \( f \) is parallel if and only if the shape operator \( AV \) of \( M \) satisfies

\[ AV, tV = -AV, tV \tag{28} \]

for any \( U, V \in \Gamma(T^\perp M) \).

**Proof.** By virtue of (8), (17) and (21), we have

\[ g((\nabla_X f)V, U) = -g(\sigma tV, X, U) - g(FAV, X, U) \]
\[ = -g(AVtV, X) - g(AV, tU) \]
\[ = -g(AVtU + A_U tV, X), \]

for any \( X \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \). This completes the proof.
Theorem 3.10. Let $M$ be a proper pseudo-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. If tensor $f$ is parallel then $M$ totally geodesic submanifold of $\tilde{M}$.

Proof. Since $f$ is parallel, from (21), we have

$$\sigma(tV, X) + FA_v X = 0$$ (29)

for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. Applying $\phi$ to (29) and by virtue of (1) and (22), we obtain

$$0 = \phi^2 A_v X + \phi \sigma(tV, X)$$

$$= A_v X + \eta(A_v X) \xi + t\sigma(tV, X) + n\sigma(tV, X).$$

Comparing tangential component, we obtain

$$A_v X + t\sigma(tV, X) = 0.$$

On the other hand for $Z \in \Gamma(TM)$, by using (8), (3.3), (17) and (28), we get

$$g(A_v X, Z) = - g(t\sigma(tV, X), Z) = - g(\sigma(tV, X), FZ)$$

$$= - g(A_{FZ} tV, X) = - g(A_v tFZ, X).$$

Taking account of $tFZ = Z + \eta(Z) \xi - T^2 Z$, we obtain

$$g(A_v Z, X) = - g(A_v Z - \eta(Z) A_v \xi + A_v T^2 Z, X)$$

$$= - g(A_v Z, X) + g(A_v X, T^2 Z)$$

$$\implies g(T^2 A_v X, Z) = 0.$$ Hence, by applying (24), we get, $0 = g(T A_v X, T Z) = \cos^2 \theta g(A_v X, Z)$. Since $M$ is a proper pseudo-slant submanifold, we conclude that $A_v = 0$ i.e., $M$ is totally geodesic in $\tilde{M}$.

Definition 3.11. A pseudo-slant submanifold $M$ of a LP-cosymplectic manifold $\tilde{M}$ is said to be $D^\theta$-geodesic(respectively $D^\perp$-geodesic) if $\sigma(X, Y) = 0$ for $X, Y \in \Gamma(D^\theta)$ (respectively $\sigma(Z, W) = 0$ for $Z, W \in \Gamma(D^\perp)$). If $\sigma(X, Z) = 0$, $M$ is called mixed geodesic submanifold for any $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$

Theorem 3.12. Let $M$ be a proper pseudo-slant submanifold of a LP-cosymplectic manifold $\tilde{M}$. If $t$ is parallel, then either $M$ is a mixed geodesic or a totally real submanifold.

Proof. By virtue of (19) and from Theorem (3.4), we obtain $f\sigma(X, Z) = 0$, for any $X \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. Also by using (19) and (22), we conclude that $f\sigma(Y, T X) - \sigma(Y, T^2 X) = - \cos^2 \theta \sigma(X, Y) = 0$. This proves our assertion.

References


Common Fixed Point of Three Contractive Type Mappings

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Abstract: In this paper we prove common fixed point theorems for sequentially convergent Kannan and Chatterjea type mappings, which are generalization of many common fixed point theorems.

Keywords: Fixed point, Affine, Sequentially convergent, Compact.

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1. Introduction

This paper aims to prove unique common fixed point theorems for three contractive type mappings on a complete metric space which extends the theorems of [1] and [2] for single mapping and [3, 4, 7] results for two mappings. Let (X,d) be a complete metric space, T : X → X be continuous, injective and sequentially convergent mapping and let S₁, S₂ be self maps of X. The authors of [7], 2016 proved that, if T, S₁, S₂ satisfies d(TS₁x, TS₂y) ≤ α(d(Tx, TS₁x) + d(Ty, TS₂y)) + βd(Tx, Ty), for all x, y ∈ X, where α > 0, β ≥ 0 such that 2α + β < 1 then S₁ and S₂ have a unique common fixed point. Also they proved, if T, S₁, S₂ satisfies d(TS₁x, TS₂y) ≤ α(d(Tx, TS₂y) + d(Ty, TS₁x)) + βd(Tx, Ty), then S₁ and S₂ have a unique common fixed point. We establish the results for the existence of unique common fixed points for three contractive mappings T, S₁, S₂ by assuming that d(Tx, S₁y) ≤ d(x, y) (or) d(Tx, S₂y) ≤ d(x, y) for all x, y ∈ X. We also prove results showing the unique common fixed point for the self maps T, S₁, S₂ on a non-empty compact subset K of a metric space (X,d), by relaxing the condition of sequentially convergent on T. In a non-empty compact convex subset K of a Banach space X, we assume that T to be affine instead of sequentially convergent and ∥Tx − S₁y∥ ≤ ∥x − y∥ or ∥Tx − S₂y∥ ≤ ∥x − y∥ for all x, y ∈ K, for the common fixed point of T with S₁, S₂.

2. Preliminaries

Definition 2.1 ([6]). Let (X,d) be a metric space. A mapping T : X → X is said to be sequentially convergent if for each sequence {yₙ} in X, the sequence {Tyₙ} converges ⇒ {yₙ} is convergent.

Definition 2.2. Let K be a non-empty subset of a Banach space X. A map T : K → K is said to be affine if T((1−λ)x+λy) = (1−λ)Tx + λTy for all x, y ∈ K and λ ∈ (0,1).

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Theorem 2.3 ([7]). Let \((X,d)\) be a complete metric space, \(T : X \to X\) be continuous, injective and sequentially convergent mapping and \(S_1, S_2 : X \to X\). If there exist \(\alpha > 0, \beta \geq 0\) such that \(2\alpha + \beta < 1\) and

\[
d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_1x) + d(Ty, TS_2y)) + \beta d(Tx, Ty),
\]

for all \(x, y \in X\), then \(S_1\) and \(S_2\) have a unique common fixed point.

Theorem 2.4 ([7]). Let \((X,d)\) be a complete metric space, \(T : X \to X\) be continuous, injective and sequentially convergent mapping and \(S_1, S_2 : X \to X\). If there exist \(\alpha > 0, \beta \geq 0\) so that \(2\alpha + \beta < 1\) and

\[
d(TS_1x, TS_2y) \leq \alpha(d(Tx, TS_2y) + d(Ty, TS_1x)) + \beta d(Tx, Ty),
\]

for all \(x, y \in X\), then \(S_1\) and \(S_2\) have a unique common fixed point.

3. Main Results

Since \(\alpha > 0\) in Theorem 2.3, The following theorem is not the corollary of the Theorem 2.3.

**Theorem 3.1.** Let \((X,d)\) be a complete metric space and let \(T : X \to X\) be continuous, injective, sequentially convergent mapping. Let \(S_1, S_2 : X \to X\) be self maps such that \(d(TS_1x, TS_2y) \leq ad(Tx, TS_1x) + bd(Ty, TS_2y) + cd(Tx, Ty)\), where \(a, b, c \in [0, 1)\) with \(a + b + c < 1\) and \(d(Tx, S_1y) \leq d(x, y)\) (or) \(d(Tx, S_2y) \leq d(x, y)\) for all \(x, y \in X\), then \(T, S_1\) and \(S_2\) have a unique common fixed point.

**Proof.** Let \(x_0 \in X\), Define \(x_n\) by \(x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1}\) for \(n = 0, 1, 2, \ldots\)

Let \(n\) be even.

\[
d(Tx_n, Tx_{n+1}) = d(TS_2x_{n-1}, TS_1x_n)
\]
\[
\leq ad(Tx_{n-1}, TS_2x_{n-1}) + bd(Tx_n, TS_1x_n) + cd(Tx_{n-1}, Tx_n)
\]
\[
= ad(Tx_{n-1}, Tx_n) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n)
\]
\[
d(Tx_n, Tx_{n+1}) \leq \frac{a + c}{1 - b} d(Tx_{n-1}, Tx_n)
\]

Since \(a + b + c < 1\), hence \(\{Tx_n\}\) is a Cauchy sequence in \(X\). Therefore \(\{Tx_n\}\) is convergent in \(X\). Since \(T\) is sequentially convergent, \(\{x_n\}\) is convergent. (i.e) \(\exists x \in X\) such that \(x_n \to x\) as \(n \to \infty\). Since \(T\) is continuous, \(Tx_n \to Tx\) as \(n \to \infty\).

Now

\[
d(Tx, TS_1x) \leq d(Tx, Tx_{2n}) + d(Tx_{2n}, TS_1x)
\]
\[
= d(Tx, Tx_{2n}) + d(TS_2x_{2n-1}, TS_1x)
\]
\[
\leq d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, TS_2x_{2n-1}) + bd(Tx, TS_1x) + cd(Tx_{2n-1}, Tx)
\]
\[
= d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, Tx_{2n}) + bd(Tx, TS_1x) + cd(Tx_{2n-1}, Tx)
\]
\[
\to bd(Tx, TS_1x) \text{ as } n \to \infty.
\]
\[ Tx = TS_1 x, \] Since \( T \) is injective, \( x = S_1 x \). Similarly \( x = S_2 x \). Hence \( x = S_1 x = S_2 x \) and
\[
d(x, Tx) \leq d(x, x_{2n}) + d(x_{2n}, Tx)
\]
\[
= d(x, x_{2n}) + d(S_2 x_{2n-1}, Tx)
\]
\[
\leq d(x, x_{2n}) + d(x_{2n-1}, x)
\]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]

\[ \therefore Tx = x. \] Hence \( T, S_1, S_2 \) have a common fixed point.

**Uniqueness:** Suppose \( \exists y \in X \) such that \( S_1 y = S_2 y = Ty = y \). Now
\[
d(Tx, Ty) = d(TS_1 x, TS_2 y)
\]
\[
\leq ad(Tx, TS_1 x) + bd(Ty, TS_2 y) + cd(Tx, Ty)
\]
\[
= ad(Tx, Tx) + bd(Ty, Ty) + cd(Tx, Ty)
\]
Since \( c < 1 \), \( Tx = Ty \) and hence \( x = y \). \( \square \)

**Corollary 3.2.** Let \( (X, d) \) be a complete metric space and let \( T : X \rightarrow X \) be continuous, injective, sequentially convergent mapping. Let \( S_1, S_2 : X \rightarrow X \) be self maps such that \( d(TS_1 x, TS_2 y) \leq \alpha d(Tx, Ty) \), where \( \alpha \in [0, 1) \) and \( d(Tx, S_1 y) \leq d(x, y) \) (or) \( d(Tx, S_2 y) \leq d(x, y) \) for all \( x, y \in X \), then \( T, S_1 \) and \( S_2 \) have a unique common fixed point.

**Proof.** The proof of the corollary follows from the above theorem by putting \( a = b = 0 \) and \( c = \alpha \). \( \square \)

**Theorem 3.3.** Let \( K \) be a non-empty compact subset of a metric space \( (X, d) \). Let \( T : K \rightarrow K \) be continuous, injective mapping and let \( S_1, S_2 \) be self mappings of \( K \). If there exist \( a \in [0, 1) \) and \( b \geq 0 \) such that \( 2a + b \leq 1 \) and \( d(TS_1 x, TS_2 y) \leq a[d(Tx, TS_1 x) + d(Ty, TS_2 y)] + bd(Tx, Ty) \), and \( d(Tx, S_1 y) \leq d(x, y) \) (or) \( d(Tx, S_2 y) \leq d(x, y) \) for all \( x, y \in K \), then \( T, S_1 \) and \( S_2 \) have a common fixed point. Further if \( b < 1 \) then \( T, S_1 \) and \( S_2 \) have a unique common fixed point.

**Proof.** For each \( n \in N \), let \( x_{2n+1} = S_1 x_{2n}, x_{2n+2} = S_2 x_{2n+1} \). Then the sequence \( \{x_n\} \subseteq K \). Since \( K \) is compact, \( \{x_n\} \) has a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \rightarrow x \) as \( k \rightarrow \infty \). Therefore \( Tx_{n_k} \rightarrow Tx \). Now
\[
d(Tx, TS_1 x) \leq d(Tx, x_{2n_k}) + d(x_{2n_k}, TS_1 x)
\]
\[
= d(Tx, x_{2n_k}) + d(TS_2 x_{2n_k-1}, TS_1 x)
\]
\[
\leq d(Tx, x_{2n_k}) + a[d(Tx, TS_1 x) + d(Ty, TS_2 y)] + bd(Tx, Ty)
\]
\[
= d(Tx, x_{2n_k}) + a[d(Tx, TS_1 x) + d(x, y)] + bd(Tx, Ty)
\]
\[ \rightarrow ad(Tx, TS_1 x) \text{ as } k \rightarrow \infty. \]

\[ \therefore Tx = TS_1 x, \] Since \( T \) is injective, \( x = S_1 x \). Similarly \( x = S_2 x \). Hence \( x = S_1 x = S_2 x \).
\[
d(x, Tx) \leq d(x, x_{2n}) + d(x_{2n}, Tx)
\]
\[
= d(x, x_{2n}) + d(S_2 x_{2n-1}, Tx)
\]
\[
\leq d(x, x_{2n}) + d(x_{2n-1}, x)
\]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]
\[ \therefore Tx = x. \text{ Hence } T, S_1, S_2 \text{ have a common fixed point.} \]

**Uniqueness:** Let \( y \) be an element in \( K \) such that \( S_1y = S_2y = Ty = y \). Now

\[
d(Tx, Ty) = d(TS_1x, TS_2y) \\
\leq a[d(Tx, TS_1x) + d(Ty, TS_2y)] + bd(Tx, Ty) \\
= a[d(Tx, Tx) + d(Ty, Ty)] + bd(Tx, Ty)
\]

If \( b < 1, Tx = Ty \) and hence \( x = y \). \( \square \)

Since \( \alpha > 0 \) in Theorem 2.4, The following theorem is not the corollary of the Theorem 2.4.

**Theorem 3.4.** Let \((X, d)\) be a complete metric space and let \( T : X \to X \) be continuous, injective, sequentially convergent mapping. Let \( S_1, S_2 : X \to X \) be self maps such that \( d(TS_1x, TS_2y) \leq ad(Tx, TS_2y) + bd(Ty, TS_1x) + cd(Tx, Ty) \), where \( a, b, c \in [0, 1) \) with \( 2a + b + c < 1 \) and \( d(Tx, S_1y) \leq d(x, y) \) (or) \( d(Tx, S_2y) \leq d(x, y) \) for all \( x, y \in X \), then \( T, S_1 \) and \( S_2 \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \), define \( x_n \) by \( x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1} \) for \( n = 0, 1, 2, \ldots \). Let \( n \) be even.

\[
d(Tx_n, Tx_{n+1}) = d(TS_2x_{n-1}, TS_1x_n) \\
\leq ad(Tx_{n-1}, TS_1x_n) + bd(Tx_n, TS_2x_{n-1}) + cd(Tx_{n-1}, Tx_n) \\
= ad(Tx_{n-1}, Tx_{n+1}) + bd(Tx_n, Tx_n) + cd(Tx_{n-1}, Tx_n) \\
\leq a[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] + cd(Tx_{n-1}, Tx_n)
\]

\[
d(Tx_n, Tx_{n+1}) \leq \frac{a + c}{1 - a} d(Tx_{n-1}, Tx_n)
\]

Since \( 2a + b + c < 1 \), hence \( \{Tx_n\} \) is a Cauchy sequence in \( X \). Therefore \( \{ Tx_n \} \) is convergent in \( X \). Since \( T \) is sequentially convergent, \( \{x_n\} \) is convergent. (i.e) \( \exists \ x \in X \) such that \( x_n \to x \) as \( n \to \infty \). Since \( T \) is continuous, \( Tx_n \to Tx \) as \( n \to \infty \). Now

\[
d(Tx, TS_1x) \leq d(Tx, Tx_{2n}) + d(Tx_{2n}, TS_1x) \\
= d(Tx, Tx_{2n}) + d(TS_2x_{2n-1}, TS_1x) \\
\leq d(Tx, Tx_{2n}) + ad(Tx_{2n-1}, TS_1x) + bd(Tx, TS_2x_{2n-1}) + cd(Tx_{2n-1}, Tx) \\
\to ad(Tx, TS_1x) \text{ as } n \to \infty.
\]

Since \( a < 1, d(Tx, TS_1x) = 0 \) implies \( Tx = TS_1x, x = S_1x \), similarly \( x = S_2x \) and

\[
d(x, Tx) \leq d(x, x_{2n}) + d(x_{2n}, Tx) \\
= d(x, x_{2n}) + d(S_2x_{2n-1}, Tx) \\
\leq d(x, x_{2n}) + d(x_{2n-1}, x) \\
\to 0 \text{ as } n \to \infty.
\]
The proof of the corollary follows from the above theorem by putting

**Proof.** Suppose \( y \in X \) such that \( S_1y = S_2y = Ty = y \). Now

\[
d(Tx, Ty) = d(TS_1x, TS_2y) \\
\leq ad(Tx, TS_2y) + bd(Ty, TS_1x) + cd(Tx, Ty) \\
= ad(Tx, Ty) + bd(Ty, Tx) + cd(Tx, Ty)
\]

Since \( a + b + c < 1 \), \( Tx = Ty \) and hence \( x = y \). \( \square \)

**Corollary 3.5.** Let \((X, d)\) be a complete metric space and let \( T : X \to X \) be continuous, injective, sequentially convergent mapping. Let \( S_1, S_2 : X \to X \) be self maps such that \( d(TS_1x, TS_2y) \leq ad(Tx, Ty) \), where \( a \in \{0, 1\} \) and \( d(Tx, S_1y) \leq d(x, y) \) (or) \( d(Tx, S_2y) \leq d(x, y) \) for all \( x, y \in X \), then \( T, S_1 \) and \( S_2 \) have a unique common fixed point.

**Proof.** The proof of the corollary follows from the above theorem by putting \( a = b = 0 \) and \( c = \alpha \). \( \square \)

**Theorem 3.6.** Let \( K \) be a non-empty compact subset of a metric space \((X, d)\). Let \( T : K \to K \) be continuous, injective and \( S_1, S_2 \) be self mappings of \( K \). If there exist \( a \in \{0, 1\} \) and \( b \geq 0 \) such that \( 2a + b \leq 1 \) and \( d(TS_1x, TS_2y) \leq a[d(Tx, TS_2y) + d(Ty, TS_1x)] + bd(Tx, Ty) \), and \( d(Tx, S_1y) \leq d(x, y) \) (or) \( d(Tx, S_2y) \leq d(x, y) \) for all \( x, y \in K \), then \( T, S_1 \) and \( S_2 \) have a unique common fixed point.

**Proof.** For each \( n \in N \), let \( x_{2n+1} = S_1x_{2n}, x_{2n+2} = S_2x_{2n+1} \). Then the sequence \( \{x_n\} \subseteq K \). Since \( K \) is compact, \( \{x_n\} \) has a subsequence \( \{x_{n_k}\} \) such that \( x_{n_k} \to x \) as \( k \to \infty \). Therefore \( Tx_{n_k} \to Tx \). Now

\[
d(Tx, TS_1x) \leq d(Tx, Tx_{2n_k}) + d(Tx_{2n_k}, TS_1x) \\
= d(Tx, Tx_{2n_k}) + d(TS_2x_{2n_k-1}, TS_1x) \\
\leq d(Tx, Tx_{2n_k}) + a[d(Tx_{2n_k-1}, TS_1x) + d(Tx, TS_2x_{2n_k-1})] + bd(Tx_{2n_k-1}, Tx) \\
= d(Tx, Tx_{2n_k}) + a[d(Tx_{2n_k-1}, TS_1x) + d(Tx, Tx_{2n_k-1})] + bd(Tx_{2n_k-1}, Tx) \\
\to ad(Tx, TS_1x) \text{ as } k \to \infty.
\]

\( \therefore Tx = TS_1x \), Since \( T \) is injective, \( x = S_1x \). Similarly \( x = S_2x \). Hence \( x = S_1x = S_2x \).

\[
d(x, Tx) \leq d(x, x_{2n+1}) + d(x_{2n+1}, Tx) \\
= d(x, x_{2n+1}) + d(S_1x_{2n}, Tx) \\
\leq d(x, x_{2n+1}) + d(x_{2n}, x) \\
\to 0 \text{ as } n \to \infty.
\]

\( \therefore Tx = x. \) Hence \( T, S_1, S_2 \) have a common fixed point.

**Uniqueness:** Let \( y \) be an element in \( K \) such that \( S_1y = S_2y = Ty = y \). Now

\[
d(Tx, Ty) = d(TS_1x, TS_2y) \\
\leq a[d(Tx, TS_2y) + d(Ty, TS_1x)] + bd(Tx, Ty) \\
= a[d(Tx, Ty) + d(Ty, Tx)] + bd(Tx, Ty) \\
(1 - (2a + b))d(Tx, Ty) \leq 0
\]

If \( 2a + b < 1 \), \( Tx = Ty \) and hence \( x = y \). \( \square \)
Common Fixed Point of Three Contractive Type Mappings

**Theorem 3.7.** Let $K$ be a non-empty compact convex subset of a Banach space $X$. Let $T : K \to K$ be continuous, injective, affine and $S_1, S_2$ be self mappings of $K$. If there exist $\alpha \in [0, 1)$ such that $\|TS_1x - TS_2y\| \leq \alpha\|Tx - Ty\|$ and $\|Tx - S_1y\| \leq \|x - y\|$ or $\|Tx - S_2y\| \leq \|x - y\|$ for all $x, y \in K$. Then $T, S_1, S_2$ have a common fixed point.

**Proof.** Let $x_0 \in K, \alpha_n \in (0, 1)$ such that $\alpha_n \to 1$, as $n \to \infty$. Define $S_{1n}, S_{2n} : K \to K$ by $S_{1n}(x) = (1 - \alpha_n)x_0 + \alpha_nS_1x, S_{2n}(x) = (1 - \alpha_n)x_0 + \alpha_nS_2x$. Then $\|TS_{1n}x - TS_{2n}y\| = \alpha_n\|TS_1x - TS_2y\| \leq \alpha_n\alpha\|Tx - Ty\|$. Then by Corollary 3.2, $S_{1n}, S_{2n}$ have a common fixed point. Let $S_{1n}(x_n) = S_{2n}(x_n) = x_n, \forall n$. Since $X$ is compact, $\{x_n\}$ has a subsequence $\{x_{nk}\}$ such that $x_{nk} \to x$ as $k \to \infty$. Therefore $T x_{nk} \to T x$. Now, $x_{nk} = S_{1nk}x_{nk} = (1 - \alpha_{nk})x_0 + \alpha_{nk}S_1x_{nk}, S_1x_{nk} \to x$ as $k \to \infty$. Similarly $S_2x_{nk} \to x$.

\[
\|Tx - TS_1x\| \leq \|Tx - TS_2x_{nk}\| + \|TS_2x_{nk} - TS_1x\| \\
\leq \|Tx - TS_2x_{nk}\| + \alpha\| Tx_{nk} - Tx \| \\
\to 0 \text{ as } k \to \infty.
\]

Hence $\|Tx - TS_1x\| = 0$. Since $T$ is injective, $x = S_1x$. Similarly $x = S_2x$. Now

\[
\|x - Tx\| \leq \|x - S_1x_{nk}\| + \|S_1x_{nk} - Tx\| \\
= \|x - S_1x_{nk}\| + \|x_{nk} - x\| \\
\to 0 \text{ as } k \to \infty.
\]

Hence $x = Tx$. Thus $T, S_1, S_2$ have a common fixed point. \hfill \Box

**References**


Geometric Approach for Banach Space Using Hausdorff Distance

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Abstract: In this paper, we study the normed linear spaces which are induced by Hausdorff distance. Barich proved the completeness of Hausdorff metric space [3]. We extend his work for the completeness of the normed linear spaces called Banach spaces, which are induced by Hausdorff distance and we proved convex Hausdorff metric space is Banach space.

Keywords: Convex set, Banach Space, Hausdorff Distance, Hausdorff Metric Space.

1. Introduction

Let’s turn the clock ahead to 1922 and given all brief discussion of the contribution of Eduard Helly, Hans Hahn and the great Polish mathematician Banach. While Eduard Helly and Hans Hahn are important players in the story of Functional analysis, making several important contributions to its early development it was Banach who gave the first complete treatment of abstract normed vector space and its the word complete that must be emphasized! in his thesis [6]. Banach discussed several important applications of theory of functionals in his own words. Ofcourse, most of us are familiar with the notion of a Banach space, which was introduced in its fully glory that is in Banach thesis. We discuss the normed linear spaces which are induced by Hausdorff distance and some of the basic concepts from Functional analysis. In a nutshell Functional analysis is a study of normed vector spaces and bounded linear operators. Thus it merges the subjects of linear algebra with the points set topology. The topologies that appears in Functional analysis will arise from Hausdorff metric space.

The geometry that follows from these consideration gives a specified approach to Banach space. Considering the above concepts, we have presented a geometric setup that allows us to obtain structure for the existence of an Banach space. Moreover, our geometric frame-work provides that generate a new setup that might be useful to determine conditions that generate the study of functionals for which some interesting results concerning the existence. In this paper, we construct the Hausdorff metric space, is to geometrize the Banach space completely. First Hausdorff distance has been considered, thus by a specific distance, largest length is calculated & properties of this metric function are studied. Finally, convex Hausdorff metric space are considered as complete normed linear space i.e., Banach space.

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2. Preliminaries

The concepts in this section should be familiar to anyone who has taken a course in real analysis. Therefore, we expect the reader to be familiar with the following definitions when applied to the metric space \((R, d)\), where \(d(x, y) = |x - y|\). However, with the exclusion of some examples, for the majority of this paper we will be working in a general metric space. Thus our definitions will be given with respect to any metric space \((X, d)\).

**Definition 2.1.** Metric space \((X, d)\) consists of a set \(X\) and a function \(d : X \times X \to R\) that satisfies the following four properties.

1. \(d(x, y) \geq 0\) for all \(x, y \in X\).
2. \(d(x, y) = 0\) if and only if \(x = y\).
3. \(d(x, y) = d(y, x)\) for all \(x, y \in X\).
4. \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

The function \(d\), which gives the distance between two points in \(X\), is called a metric.

**Definition 2.2.** Let \(v \in X\) and let \(r > 0\). Open ball centered at \(v\) with radius \(r\) is defined by \(B_d(v, r) = \{x \in X : d(x, v) < r\}\).

**Definition 2.3.** A set \(E \subseteq X\) is Bounded in \((X, d)\) if there exist \(x \in X\) and \(M > 0\) such that \(E \subseteq B_d(x, M)\).

**Definition 2.4.** A set \(K \subseteq X\) is Totally bounded if for each \(\epsilon > 0\) there is a finite subset \(\{x_i : 1 \leq i \leq n\}\) of \(K\) such that \(K \subseteq \bigcup_{i=2}^n B_d(x_i, \epsilon)\).

For the following definitions, let \(\{x_n\}\) be a sequence in a metric space \((X, d)\).

**Definition 2.5.** The sequence \(\{x_n\}\) Converges to \(x \in X\) if for each \(\epsilon > 0\) there exists a positive integer \(N\) such that \(d(x_n, x) < \epsilon\), for all \(n \geq N\). We say \(\{x_n\}\) converges if there exists a point \(x \in X\) such that \(\{x_n\}\) converges to \(x\).

**Definition 2.6.** The sequence \(\{x_n\}\) is a Cauchy sequence if for each \(\epsilon > 0\) there exists a positive integer \(N\) such that \(d(x_n, x_m) < \epsilon\) for all \(m, n \geq N\).

**Definition 2.7.** A metric space \((X, d)\) is Complete if every Cauchy sequence in \((X, d)\) converges to a point in \(X\).

**Definition 2.8.** A set \(K \subseteq X\) is Sequentially compact in \((X, d)\) if each sequence in \(K\) has a subsequence that converges to a point in \(K\).

**Definition 2.9.** Norm \(\|\cdot\|\) on a linear space \(X\) is a mapping \(X \to R\) satisfying

1. \(\|x\| \geq 0\) for all \(x \in X\).
2. \(\|x\| = 0\) if and only if \(x = 0\).
3. \(\|\lambda x\| = |\lambda|\|x\|\) for all \(\lambda \in R\) and \(x \in X\).
4. (Triangle inequality) \(\|x + y\| \leq \|x\| + \|y\|\) for all \(x, y \in X\).

A normed linear space \((X, \|\cdot\|)\) is a linear space \(X\) equipped with a norm \(\|\cdot\|\).

**Definition 2.10.** A complete normed linear space is called a Banach space.
Corollary 2.11 ([5]). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in a metric space \((X,d)\). If \(\{x_n\}\) converges to \(x\) and \(\{y_n\}\) converges to \(y\), then \(d(x_n,y_n)\) converges to \(d(x,y)\).

Corollary 2.12 ([5]). If \(\{z_k\}\) is a sequence in a metric space \((X,d)\) with the property that \(d(z_k, z_{k+1}) < \frac{1}{2^k}\) for all \(k\), then \(\{z_k\}\) is a Cauchy sequence.

Lemma 2.13. Let \((X,d)\) be a metric space and let \(A\) be a closed subset of \(X\). If \(\{a_n\}\) converges to \(x\) and \(a_n \in A\) for all \(n\), then \(x \in A\).

Proof. Suppose \(\{a_n\}\) is a sequence that converges to \(x\) and \(a_n \in A\) for all \(n\). There are two cases to consider. If there exists a positive integer \(n\) such that \(a_n = x\), then it is clear \(x \in A\). If there does not exist a positive integer \(n\) such that \(a_n = x\), then \(x\) is a limit point of \(A\) by Theorem 8.49 in [5]. Since \(A\) is closed, \(x \in A\).

3. Construction of the Hausdorff Metric

We now define the Hausdorff metric on the set of all nonempty, compact subsets of a metric space. Let \((X,d)\) be a complete metric space and let \(\kappa\) be the collection of all nonempty compact subsets of \(X\). Note that \(\kappa\) is closed under finite union and nonempty intersection. For \(x \in X\) and \(A,B \in \kappa\), define

\[
r(x,B) = \inf\{d(x,b) : b \in B\} \quad \text{and} \quad \rho(A,B) = \sup\{r(a,B) : a \in A\}.
\]

Note that \(r\) is nonnegative and exists by the completeness axiom, since \(d(a,b) \geq 0\) by the definition of a metric space. Since \(r\) exists and is nonnegative, then both \(\rho(A,B)\) and \(\rho(B,A)\) exist and are nonnegative. In addition, we define the Hausdorff distance between sets \(A\) and \(B\) in \(\kappa\) as

\[
h(A,B) = \max\{\rho(A,B),\rho(B,A)\}.
\]

Before proving that \(h\) defines a metric on the set \(\kappa\), let us consider a few examples to get a grasp on how these distances work. Consider the following example of closed interval sets in \((R,d)\), where \(d(x,y) = |x - y|\).

Example 3.1. Let \(A = [0,10]\) and let \(B = [12,21]\). We find that \(r(x,B)\) is going to be the infimum of the set of distances from each \(a \in A\) to the closest point in \(B\). As an example of one of these distances, consider \(a = 2\). Then \(r(2,B) = \inf\{d(2,b) : b \in B\} = d(2,12) = 10\). We can note that for each \(a \in A\), the closest point in \(B\) that gives the smallest distance will always be \(b = 12\). Therefore, we find that \(\rho(A,b) = \sup\{d(a,12) : a \in A\}\). The point \(a = 0\) in \(A\) maximizes this distance.

Therefore \(\rho(A,B) = d(0,12) = |12 - 0| = 12\).

Similarly, we find that \(\rho(B,A) = \sup\{d(b,10) : b \in B\}\), since the point \(a = 10\) will give the smallest distance to any point in \(B\). The point \(b = 21\) in \(B\) maximizes this distance, so we have \(\rho(B,A) = d(10,21) = |10 - 21| = 11\). It follows that \(h(A,B) = \max\{\rho(A,B),\rho(B,A)\} = 12\).

Now that we have gained a knowledge on how \(r, \rho, \) and \(h\) work in a few special cases, we refer some basic properties of \(r\) and \(\rho\).

Theorem 3.2 ([3]). Let \(x \in X\) and let \(A,B,C \in \kappa\).

\(\langle 1 \rangle\). \(r(x,A) = 0\) if and only if \(x \in A\).

\(\langle 2 \rangle\). \(\rho(A,B) = 0\) if and only if \(A \subseteq B\).

\(\langle 3 \rangle\). There exists \(a_x \in A\) such that \(r(x,A) = d(x,a_x)\).
There exists \( a^* \in A \) and \( b^* \in B \) such that \( \rho(A, B) = d(a^*, b^*) \).

If \( A \subseteq B \), then \( r(x, B) \leq r(x, A) \).

If \( B \subseteq C \), then \( \rho(A, C) \leq \rho(A, B) \).

\( \rho(A \cup B, C) = \max \{ \rho(A, C), \rho(B, C) \} \).

\( \rho(A, B) \leq \rho(A, C) + \rho(C, B) \).

4. Hausdorff Metric Space

Normed linear space is a Hausdorff metric space equipped with the metric \( d(x, y) = \|x - y\| \). A metric in a linear space defines a norm if it satisfies translational invariant \( d(x - z, y - z) = d(x, y) \) and homogeneity \( d(\lambda x, 0) = \lambda d(x, 0) \). Given a complete metric space \((X, d)\), we have now construction of new metric space \((\kappa, h)\) from the nonempty, compact subsets of \( X \) using the Hausdorff distance. The following theorem shows Hausdorff distance defines a metric on \( \kappa \).

**Theorem 4.1** ([3]). The set \( \kappa \) with the Hausdorff distance \( h \) define a metric space \((\kappa, h)\).

**Proof.** To prove that \((\kappa, h)\) is a metric space, we need to verify the following four properties.

1. \( h(A, B) \geq 0 \) for all \( A, B \in \kappa \).
2. \( h(A, B) = 0 \) if and only if \( A = B \).
3. \( h(A, B) = h(B, A) \) for all \( A, B \in \kappa \).
4. \( h(A, B) \leq h(A, C) + h(C, B) \) for all \( A, B, C \in \kappa \).

To prove the first property, since \( \rho(A, B) \) and \( \rho(B, A) \) are nonnegative, it follows that \( h(A, B) \geq 0 \) for all \( A, B \in \kappa \).

For the second property, suppose \( A = B \). Therefore \( A \subseteq B \) and \( B \subseteq A \). By Property (2) of Theorem 2.4 we find that \( \rho(A, B) \) and \( \rho(B, A) = 0 \), and thus \( h(A, B) = 0 \). Now suppose \( h(A, B) = 0 \). This implies \( \rho(A, B) = \rho(B, A) = 0 \). By property (2) of Theorem 3.2, we see that \( A \subseteq B \) and \( B \subseteq A \) and it follows that \( A = B \).

The third property can be proved from the symmetry of the definition since

\[
\begin{align*}
  h(A, B) &= \max \{ \rho(A, B), \rho(B, A) \} \\
  &= \max \{ \rho(B, A), \rho(A, B) \} \\
  &= h(B, A).
\end{align*}
\]

The final property follows from the definition of \( \rho \) and \( h \) and from property (8) of Theorem 3.2. We find that

\[
\rho(A, B) \leq \rho(A, C) + \rho(C, B)
\]

Similarly,

\[
\rho(B, A) \leq \rho(B, C) + \rho(C, A)
\]

Therefore, \( h(A, B) = \max \{ \rho(A, B), \rho(B, A) \} \leq h(A, C) + h(C, B) \).
Therefore we know that $h$ defines a metric on $\kappa$. Hence it defines Hausdorff metric space $(\kappa, h)$. In the next section, we will look at example of what this metric space might look like, and then one may proceed to prove if the metric space $(X, d)$ is complete, then the metric space $(\kappa, h)$ which is induced by Hausdorff distance is also complete.

**Example 4.2.** Let $(\mathbb{R}, d_0)$ be the complete metric space, where $d_0$ is the discrete metric,

$$d_0(x, y) = \begin{cases} 
0, & \text{when } x = y. \\
1, & \text{when } x \neq y.
\end{cases}$$

Since $\kappa$ is the set of all nonempty, compact subsets of $(\mathbb{R}, d_0)$, we find that $\kappa$ is the set of all nonempty finite subsets of $\mathbb{R}$. The infinite sets are not in $\kappa$ because they are not totally bounded and are thus not compact. Furthermore, we may notice that

$$r(x, B) = \inf\{d_0(x, b) : b \in B\} = d_0(x, y) = \begin{cases} 
0, & \text{when } x \in B. \\
1, & \text{when } x \notin B.
\end{cases}$$

Therefore,

$$\rho(A, B) = \sup\{r(a, B) : a \in A\} = \begin{cases} 
0, & \text{when } a \in B. \\
1, & \text{when } a \notin B.
\end{cases}$$

So it follows that

$$h(A, B) = \begin{cases} 
0, & \text{when } A = B. \\
1, & \text{when } A \neq B.
\end{cases}$$

Therefore we have a metric space with the set $\kappa$ of the discrete subsets of $\mathbb{R}$ with the Hausdorff metric as the discrete metric. It is easy to verify that our newly created space is not totally bounded. However, we know all discrete metric spaces are complete, so $(\kappa, h)$ is complete. Therefore, the space $(\kappa, h)$ of finite sets with the discrete metric is an example of our Hausdorff induced metric space $(\kappa, h)$.

To illustrate our notion of completeness, now briefly consider a sequence of nonempty compact sets that converges to the unit circle in $\mathbb{R}^2$. This is an example a converging Cauchy sequence in the Hausdorff induced metric space that converges to a set also in the space.

### 5. Proving that the Hausdorff Metric Space $(\kappa, h)$ is Complete

As previously stated, to be a complete metric space, every Cauchy sequence in $(\kappa, h)$ must converge to a point in $\kappa$. Therefore, in order to prove that the metric space $(\kappa, h)$ is complete, we will choose an arbitrary Cauchy sequence $\{A_n\}$ in $\kappa$ and show that it converges to some $A \in \kappa$. Define $A$ to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to $x$ and satisfies $x_n \in A_n$ for all $n$. We will eventually show that the set $A$ is an appropriate candidate. However, we must begin with some important theorems regarding $A$. Given a set $A \in \kappa$ and a positive number $\epsilon$, we define the set $A + \epsilon$ by $\{x \in X : r(x, A) \leq \epsilon\}$. We need to show that this set is closed for all possible choices of $A$ and $\epsilon$. To do this, we will begin by choosing an arbitrary limit point of the set, $A + \epsilon$, and then showing that it is contained in the set.

**Proposition 5.1.** $A + \epsilon$ is closed for all possible choices of $A \in \kappa$ and $\epsilon > 0$.

However, the following theorem gives us an alternative way of proving convergence.

**Theorem 5.2 ([3]).** Suppose that $A, B \in \kappa$ and that $\epsilon > 0$. Then $h(A, B) \leq \epsilon$ if and only if $A \subseteq B + \epsilon$ and $B \subseteq A + \epsilon$. 
Extension Lemma: Let \( \{A_n\} \) be a Cauchy sequence in \( \kappa \) and let \( \{n_k\} \) be an increasing sequence of positive integers. If \( \{x_{n_k}\} \) is a Cauchy sequence in \( X \) for which \( x_{n_k} \in A_{n_k} \) for all \( k \), then there exists a Cauchy sequence \( \{y_n\} \) in \( X \) such that \( y_n \in A_n \) for all \( n \) and \( y_{n_k} = x_{n_k} \) for all \( k \).

The following lemma makes use of the extension lemma to guarantee that \( A \) is closed and nonempty. We will need this fact in proving that \( A \) is in \( \kappa \), since we must show that \( A \) is a nonempty, compact subset of \( \kappa \). This lemma gives us that \( A \) is closed and nonempty. Since closed and totally bounded sets are compact, it remains to show that \( A \) is totally bounded.

**Lemma 5.3 ([5]).** Let \( \{A_n\} \) be a sequence in \( \kappa \) and let \( A \) be the set of all points \( x \in X \) such that there is a sequence \( \{x_n\} \) that converges to \( x \) and satisfies \( x_n \in A_n \) for all \( n \). If \( \{A_n\} \) is a Cauchy sequence, then the set \( A \) is closed and nonempty.

With the previous lemma, to prove \( A \in \kappa \), it only remains to show that \( A \) is totally bounded. The following lemma will allow us to do so.

**Lemma 5.4 ([5]).** Let \( \{D_n\} \) be a sequence of totally bounded sets in \( X \) and let \( A \) be any subset of \( X \). If for each \( \epsilon > 0 \), there exists a positive integer \( N \) such that \( A \subseteq D_N + \epsilon \), then \( A \) is totally bounded.

It gives the foundation to prove complete metric space \((X, d)\), we constructed the metric space \((\kappa, h)\) from the nonempty compact subsets of \( X \) using the Hausdorff metric. After examining important theorems and results, we can now state that

**Theorem 5.5 ([3]).** If \((X, d)\) is complete, then \((\kappa, h)\) is complete.

**Proof.** Let \( \{A_n\} \) be a Cauchy sequence in \( \kappa \), and define \( A \) to be the set of all points \( x \in X \) such that there is a sequence \( \{x_n\} \) that converges to \( x \) and satisfies \( x_n \in A_n \) for all \( n \). We must prove that \( A \in \kappa \) and \( \{A_n\} \) converges to \( A \).

By Lemma 5.3, the set \( A \) is closed and nonempty. Let \( \epsilon > 0 \). Since \( \{A_n\} \) is Cauchy sequence then there exists a positive integer \( N \) such that \( h(A_m, A_n) < \epsilon \) for all \( m, n \geq N \). \( A_m \subseteq A_n + \epsilon \) for all \( m, n \geq N \). Let \( a \in A \), then we want to show \( a \in A + \epsilon \). Fix \( n \geq N \), by definition of the set \( A \), there exists a sequence \( \{x_i\} \) such that \( x_i \in A_i \) for all \( i \) and \( \{x_i\} \) converges to \( a \). By Proposition 5.1 we know that \( A_n + \epsilon \) is closed. Since \( x_i \in A_n + \epsilon \) for each \( i \), then it follows that \( a \in A_n + \epsilon \). This shows that \( A \subseteq A_n + \epsilon \). By Lemma 5.4, the set \( A \) is totally bounded. Additionally, we know \( A \) is complete, since it is a closed subset of a complete metric space. Since \( A \) is nonempty, complete and totally bounded, then \( A \) is compact and thus \( A \in \kappa \).

Let \( \epsilon > 0 \), to show that \( \{A_n\} \) converges to \( A \in \kappa \), we need to show that there exists a positive integer \( N \) such that \( h(A_n, A) < \epsilon \) for all \( n \geq N \). To do this, we know that \( A \subseteq A_n + \epsilon \) and \( A_n \subseteq A + \epsilon \). From the first part of our proof, we know there exists \( N \) such that \( A \subseteq A_n + \epsilon \) for all \( n \geq N \).

To prove \( A_n \subseteq A + \epsilon \) let \( \epsilon > 0 \). Since \( \{A_n\} \) is a Cauchy sequence, we can choose a positive integer \( N \) such that \( h(A_m, A_n) < \frac{\epsilon}{2} \) for all \( m, n \geq N \). Since \( \{A_n\} \) is a Cauchy sequence in \( \kappa \), there exists a strictly increasing sequence \( \{n_i\} \) of positive integers such that \( n_i > N \) and such that \( h(A_m, A_n) < \epsilon 2^{-i-1} \) for all \( m, n > n_i \). We can use property (3) of Theorem 3.2 to get the following:

\[
\text{Since } A_n \subseteq A_{n_1} + \frac{\epsilon}{2} \exists x_{n_1} \in A_{n_1} \ni d(y, x_{n_1}) \leq \frac{\epsilon}{2},
\]

\[
\text{since } A_{n_1} \subseteq A_{n_2} + \frac{\epsilon}{4} \exists x_{n_2} \in A_{n_2} \ni d(x_{n_1}, x_{n_2}) \leq \frac{\epsilon}{4},
\]

\[
\text{since } A_{n_2} \subseteq A_{n_3} + \frac{\epsilon}{8} \exists x_{n_3} \in A_{n_3} \ni d(x_{n_2}, x_{n_3}) \leq \frac{\epsilon}{8}, \ldots,
\]

by continuing this process we are able to obtain a sequence \( \{x_{n_i}\} \) such that for all positive integers \( i \) then \( x_{n_i} \in A_{n_i} \) and \( d(x_{n_1}, x_{n_{i+1}}) \leq \epsilon 2^{-i-1} \). By corollary 2.12, we find \( x_{n_1} \) is a Cauchy sequence, so by the extension lemma the limit of the sequence \( a \) is in \( A \). Additionally we find that

\[
d(y, x_{n_i}) \leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + d(x_{n_2}, x_{n_3}) + \cdots + d(x_{n_{i-1}}, x_{n_i}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots + \frac{\epsilon}{2^i} < \epsilon.
\]
Since \( d(y, x_n) \leq \varepsilon \) for all \( i \), it follows that \( d(y, a) \leq \varepsilon \) and therefore \( y \in A + \varepsilon \). Thus we know that there exists \( N \) such that \( A_n \subseteq A + \varepsilon \), so it follows that \( h(A_n, A) < \varepsilon \) for all \( n \geq N \) and thus \( \{ A_n \} \) converges to \( A \in \kappa \). Therefore, if \( (X, d) \) is complete, then \( (\kappa, h) \) is complete.

6. Convex Hausdorff Metric Space as a Banach Space

Banach spaces are less special than Hilbert spaces but still sufficiently simple that their fundamental property can be explained readily several standard results which are true in greater generality have simpler and more transparent proofs in this setting. The Banach-Steinhaus uniform boundedness theorem and the Open Mapping Theorem are significantly more substantial than the first result here, since they invoke the Baire Category Theorem. Then Hahn Banach Theorem is non-trivial but does not use completeness.

Finally as made clear in work of Gelfand and Grothendieck and of many others, many subtler sorts of topological vector spaces are expressible as limit of Banach space, making clear that Banach spaces play an even more central role than would be apparent from many conventional elementary function analysis text. But, just to be on the safer side Banach introduced the axioms for vector space \( X \) (these were known at the time, but were apparently not considered well-known) and assumes that the spaces \( X \) carries a norm. Banach space named after great mathematician of twentieth century Banach and Banach space theory is presented in a broad mathematical context, using tools from such areas as set theory, topology, algebra, probability theory and logic. Equal emphasis is given to both spaces and operators. The standard notations of distance between two Banach spaces is the Banach measure distance and is given by \( d(x, y) = \| x - y \| \).

In the year 1972, A.L. Brown has studied on the subspaces of Banach space[1]. Later N.J. Kalton and M.I. Ostorski worked on distance between two Banach space in 1997[4]. Russ Gordon, worked for real analysis[5]. Katie Barich, worked on probability theory and logic. Equal emphasis is given to both spaces and operators. The standard notations of distance between two Banach spaces is the Banach measure distance and is given by \( d(x, y) = \| x - y \| \).

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Since, Barich proved the completeness of Hausdorff metric space i.e., \( X \) is complete[3]. Consider \( \| \| : X \to R \) be a normed function. In the first instance, let \( A \) and \( B \) be two nonempty sets in \( X \). By definition, \( h(A, B) = \max \{ \rho(A, B), \rho(B, A) \} \), but \( \rho(A, B) = \sup \{ r(x, B) : \forall x \in A \} \); where \( r(x, B) = \inf \{ \| x - b \| : \forall b \in B \} \) then, we have \( \| x - b \| \geq 0 \). Hence, \( h(A, B) \geq 0 \).

In the second instance, we have \( \rho(A, B) = \sup \{ r(x, B) : \forall x \in A \} \); where \( r(x, B) = \inf \{ \| x - b \| : \forall b \in B \} \) then, we have \( \| x - b \| = 0 \Rightarrow x = b \). \( x \) and \( b \) are the arbitrary elements then every element of \( A \) is element of \( B \). \( \Rightarrow A \subseteq B \). Similarly, \( B \subseteq A \). Then we have \( A = B \).

Conversely, If \( A = B \), \( \| x - b \| = 0 \forall x \in A \) \& \( b \in B \):

We have \( r(x, B) = \inf \{ \| x - b \| : \forall b \in B \} = 0 \Rightarrow \rho(A, B) = \sup \{ r(x, B) : \forall x \in A \} = 0 \); Then we have \( h(A, B) = 0 \). Hence \( h(A, B) = 0 \) iff \( A = B \). The third instance we have triangular inequality. By definition, \( h(A, B) = \max \{ \rho(A, B), \rho(B, A) \} \), but \( \rho(A, B) = \sup \{ r(x, B) : \forall x \in A \} \), where

\[
r(x, b) = \inf \{ \| x - b \| : \forall x \in A, b \in B \}
\]

\[
\leq \inf \{ \| x - a + a - b \| : \forall a \in C, b \in B \}
\]

\[
\leq \inf \{ \| x - a \| : \forall a \in A, a \in C \} + \inf \{ \| a - b \| : \forall a \in C, b \in B \}
\]

\[
\leq r(x, a) + r(a, b)
\]

\[
\Rightarrow \rho(A, B) \leq \rho(A, C) + \rho(C, B)
\]

\[
Hence, \quad h(A, B) \leq h(A, C) + h(C, B).
\]
In the last instance, we get $h(\alpha A, \alpha B) = |\alpha| \inf \{\|x - b\| : \forall \ x \in A, \ b \in B\} = |\alpha|h(A, B)$. Hence we state that.

**Theorem 6.1.** Let $X$ be convex Hausdorff metric space. If $X$ is normed linear space then it is Banach.

### 7. Conclusion

Before proceeding, we would like to have a short digression. During the last several decades, there have been several distance functions that have been proposed purporting to capture the collective/topological properties of systems of many degrees of freedom. One motivation for the formulation of such distance functions, such as the Hausdorff distance is to determine the topological properties(mainly Geometric structure) of systems with long-range interactions. From the Felix Hausdorff viewpoint, this distance clearly gives the images/structures in this category. Assuming that such distance functions may prove to be applicable to a path-integral formulation of aspects of semi-classical or even quantum gravity, the arguments of the present work will still hold without any major modifications. The minor modification needed in case, such distance functions are pertinent, is to use in Banach spaces whose structure has been determined. One could possibly use the Hausdorff distance subject to appropriate structure of space-time, for such a purpose. A second minor, for our purposes, modification may be to substitute some other measure in the place of the often used metric measure. Beyond these points, we expect the above analysis (Theorem 6.1). Hausdorff distance, that defines a metric on the space of all nonempty, compact subsets of the convex metric space. The main object of this paper is to measure greatest distance that we called as Hausdorff distance between complete normed linear spaces i.e., Banach spaces. Only the construction of Banach space has been considered in this paper, but this simple technique can also be tackled straightforwardly by other constructions like Hilbert space. This task is left to the reader.

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### References


On $W_9$–Curvature Tensor of Generalized Sasakian-Space-Forms

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Abstract: The object of the present paper is to study generalized Sasakian-space-forms satisfying certain curvature conditions on $W_9$– curvature tensor. In this paper, we study $W_9$– semisymmetric, $W_9$–flat, $\xi = W_9$– flat, generalized Sasakian-space-forms satisfying $I(\xi, X). \eta = 0, I(\xi, X). R = 0, I(\xi, X). P = 0$ and $I(\xi, X). C = 0$.

MSC: 53C25, 53D15.

Keywords: Generalized Sasakian-space form, $W_9$– curvature tensor, Concircular curvature tensor, Ricci tensor, $\eta$–Einstein Manifold, scalar curvature.

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1. Introduction

P. Alegre, D. Blair and A. Carriazo [9] introduced and studied generalized Sasakian-space-forms. In 2011, M.M. Tripathi and P. Gupta [7] introduced and studied $\tau$– curvature tensor in semi-Riemannian manifolds. They studied some properties of $\tau$– curvature tensor. They defined $W_9$– curvature tensor of type $(0, 4)$ for $(2n + 1)$–dimensional Riemannian manifold, as

$$W_9(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2n}\{S(X, Y)g(Z, U) - g(Y, Z)S(X, U)\}$$

(1)

where $R$ and $S$ denote the Riemannian curvature tensor of type $(0, 4)$ defined by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ and the Ricci tensor of type $(0, 2)$ respectively. The curvature tensor defined by (1) is known as $W_9$– curvature tensor. A manifold whose $W_9$– curvature tensor vanishes at every point of the manifold is called $W_9$– flat manifold. They also define $\tau$–conservative semi-Riemannian manifolds and give necessary and sufficient condition for semi-Riemannian manifolds to be $\tau$– conservative. Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that $M$ is generalized Sasakian-space-form if there exist three functions $f_1, f_2, f_3$ on $M$ such that the curvature tensor $R$ is given by

$$R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z\}$$

$$+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}$$

(2)

for any vector fields $X, Y, Z$ on $M$. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [8] the authors cited several examples of generalized Sasakian-space-forms. Alegre et al. [10] have given results on B.Y. Chen’s inequality on submanifolds.

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of generalized complex space-forms and generalized Sasakian-space-forms. Al. Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms \cite{11,12}. Sreenivasa. G.T. Venkatesha and Bagewadi C.S. \cite{13} have studied some results on \((LCS)_{2n+1}\)-Manifolds. S. K. Yadav, P.K. Dwivedi and D. Suthar \cite{14} studied \((LCS)_{2n+1}\)- Manifolds satisfying certain conditions on the concircular curvature tensor. De and Sarakar \cite{15} have studied generalized Sasakian-space-forms regarding projective curvature tensor. Motivated by the above studies, in the present paper, we study flatness and symmetry property of generalized Sasakian-space-forms regarding \(W_9\)-curvature tensor. The present paper is organized as follows:

In this paper, we study the \(W_9\)-curvature tensor of generalized Sasakian-space-forms with certain conditions. In section 2, some preliminary results are recalled. In section 3, we study \(W_9\)- semisymmetric generalized Sasakian-space-forms. Section 4 deals with \(\xi-W_9\) flat generalized Sasakian-space-forms. Generalized Sasakian-space-forms satisfying \(I.S = 0\) are studied in section 5. In section 6, \(W_9\)- flat generalized Sasakian-space-forms are studied. Section 7 is devoted to study of generalized Sasakian-space-forms satisfying \(I.R = 0\). In section 8, generalized Sasakian-space-forms satisfying \(I.P = 0\). The last section contains generalized Sasakian-space-forms satisfying \(I.C = 0\).

2. Preliminaries

An odd – dimensional differentiable manifold \(M^{2n+1}\) of differentiability class \(C^{r+1}\), there exists a vector valued real linear function \(\Phi\), a 1-form \(\eta\), associated vector field \(\xi\) and the Riemannian metric \(g\) satisfying

\[
\Phi^2(X) = -X + \eta(X)\xi, \Phi(\xi) = 0
\]  
\[\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\Phi X) = 0
\]  
\[g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]  

for arbitrary vector fields \(X\) and \(Y\), then \((M^{2n+1}, g)\) is said to be an almost contact metric manifold \cite{4}, and the structure \((\Phi, \xi, \eta, g)\) is called an almost contact metric structure to \(M^{2n+1}\). In view of (3), (4) and (5), we have

\[
g(\Phi X, Y) = -g(X, \Phi Y), g(\Phi X, X) = 0
\]  
\[\nabla_X \eta(Y) = g(\nabla_X \xi, Y)
\]

Again we know \cite{9} that in a \((2n + 1)\)- dimensional generalized Sasakian-space-form, we have

\[
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z\} + f_3(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi)
\]  

for any vector field \(X, Y, Z\) on \(M^{2n+1}\), where \(R\) denotes the curvature tensor of \(M^{2n+1}\) and \(f_1, f_2, f_3\) are smooth functions on the manifold. The Ricci tensor \(S\) and the scalar curvature \(r\) of the manifold of dimension \((2n + 1)\) are respectively, given by

\[
S(X, Y) = (2n f_1 + 3 f_2 - f_3)g(X, Y) - (3 f_2 + (2n - 1)f_3)\eta(X)\eta(Y)
\]  
\[QX = (2n f_1 + 3 f_2 - f_3)X - (3 f_2 + (2n - 1)f_3)\eta(X)\xi
\]  
\[r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3
\]
Also for a generalized Sasakian-space-forms, we have

\[ R(X, Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y) \]  
\[ R(\xi, X)Y = -R(X, \xi)Y = (f_1 - f_3)(g(X, Y)\xi - \eta(Y)X) \]  
\[ \eta(R(X, Y)Z) = (f_1 - f_3)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) \]  
\[ S(X, \xi) = 2n(f_1 - f_3)\eta(X) \]  
\[ Q\xi = 2n(f_1 - f_3)\xi \]

where \( Q \) is the Ricci Operator, i.e.

\[ g(QX, Y) = S(X, Y) \]

For a \((2n + 1)\)-dimensional \((n > 1)\) Almost Contact Metric, the \( W_9 \)-curvature tensor \( I \) is given by

\[ I(X, Y)Z = R(X, Y)Z - \frac{1}{2n} \{S(X, Y)Z - g(Y, Z)QX\} \]

The \( W_9 \)-curvature tensor \( I \) in a generalized Sasakian-space-form satisfies

\[ I(X, Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y) - \frac{1}{2n} \{(2nf_1 + 3f_2 - f_3)(g(X, Y)\xi - \eta(Y)X)\} \]  
\[ I(\xi, Y)\xi = (f_1 - f_3)\{\eta(Y)\xi - \eta(X)\xi\} \]  
\[ I(X, \xi)\xi = \frac{1}{2n} \{(4nf_1 + 3f_2 - (2n + 1)f_3)(X - \eta(X)\xi) \]  
\[ I(\xi, X)\xi = (f_1 - f_3)\{2g(X, Y)\xi - \eta(X)Y - \eta(Y)X\} \]  
\[ I(\xi, X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\} \]

Given an \((2n + 1)\)-dimensional Riemannian manifold \((M, g)\), the Concircular curvature tensor \( \tilde{C} \) is given by

\[ \tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)} \{g(Y, Z)X - g(X, Z)Y\} \]

\[ \tilde{C}(\xi, X)Y = [f_1 - f_3 - \frac{r}{2n(2n + 1)}] \{g(X, Y)\xi - \eta(Y)X\} \]

and

\[ \eta(\tilde{C}(X, Y)Z) = [f_1 - f_3 - \frac{r}{2n(2n + 1)}] \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \]

and Projective curvature tensor is given by

\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} \{S(Y, Z)X - S(X, Z)Y\} \]

and related term

\[ \eta(P(X, Y)\xi) = 0 \]
\[ \eta(P(\xi, X)Z) = \frac{1}{2n} S(X, Z) - (f_1 - f_3)g(X, Z) \]
\[ \eta(P(\xi, Y)Z) = (f_1 - f_3)g(Y, Z) - \frac{1}{2n} S(Y, Z) \]

for any vector field \( X, Y, Z \) on \( M \).
3. $W_9$—Semisymmetric Generalized Sasakian-Space-Forms

Definition 3.1. A $(2n+1)$—dimensional $(n > 1)$ generalized Sasakian-space-form is said to be $W_9$—semisymmetric if it satisfies $R.I = 0$, where $R$ is the Riemannian curvature tensor and $I$ is the $W_9$—curvature tensor of the space forms.

Theorem 3.2. A $(2n+1)$—dimensional $(n > 1)$ generalized Sasakian-space-form is $W_9$—semisymmetric if and only if $f_1 = f_3$.

Proof. Let us suppose that the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $W_9$—semisymmetric, then we have

\[ R(\xi, U)I(X, Y)\xi = 0 \]  

(29)

The above equation can be written as

\[ R(\xi, U)I(X, Y)\xi - I(R(\xi, U)X, Y)\xi - I(X, R(\xi, U)Y)\xi - I(X, Y)R(\xi, U)\xi = 0 \]  

(30)

In view of (4), (12) and (13) the above equation reduces to

\[ (f_1 - f_3)\{g(U, I(X, Y)\xi) - \eta(I(X, Y)\xi)U - g(X, U)I(\xi, Y)\xi + \eta(X)I(U, Y)\xi \]  

\[ - g(U, Y)I(X, \xi)\xi + I(X, U)\eta(Y)\xi - I(X, Y)\eta(U)\xi + I(X, Y)\} = 0 \]  

(31)

In view of (18), (19) and (20) and taking the inner product of above equation with $\xi$, we get

\[ (f_1 - f_3)\{g(U, I(X, Y)\xi) - \frac{1}{2n}(2nf_1 + 3f_2 - f_3)(-g(X, Y)\eta(U) \]  

\[ + g(U, Y)\eta(X) + g(X, U)\eta(Y) - g(X, Y)\eta(U) + \eta(I(X, Y)U)\} = 0 \]  

(32)

On solving above equation, we get

\[ \frac{1}{2n}(f_1 - f_3)\{(3f_2 + (2n - 1)f_3)(g(Y, U)\eta(X) - \eta(X)\eta(Y)\eta(U))\} = 0 \]  

(33)

From the above equation, we have either $f_1 = f_3$ or

\[ g(Y, U)\eta(X) - \eta(X)\eta(Y)\eta(U) = 0 \]  

(34)

which is not possible in generalized Sasakian-space-form. Conversely, if $f_1 = f_3$, then from (13), $R(\xi, U) = 0$. Then obviously $R.I = 0$ is satisfied. This completes the proof.

4. $\xi—W_9$—Flat Generalized Sasakian-Space-Forms

Definition 4.1. A $(2n+1)$—dimensional $(n > 1)$ generalized Sasakian-space-form is said to be $W_9$—flat [5] if $I(X, Y)\xi = 0$ for all $X, Y \in TM$.

Theorem 4.2. A $(2n+1)$—dimensional $(n > 1)$ generalized Sasakian-space-form is $\xi—W_9$—flat if and only if it is $\eta$—Einstein Manifold.
Proof. Let us consider that a generalized Sasakian-space-form is $\xi - W_9$– flat, i.e. $I(X,Y)\xi = 0$. Then in view of (18), we have

$$R(X,Y)\xi = \frac{1}{2n} \{ S(X,Y)\xi - g(Y,\xi)QX \}$$  \hspace{1cm} (35)$$

$$R(X,Y)\xi = \frac{1}{2n} \{ S(X,Y)\xi - \eta(Y)QX \}$$  \hspace{1cm} (36)$$

By using (12) and (14) above equation becomes

$$\eta(Y)QX = (2nf_1 + 3f_2 - f_3)g(X,Y)\xi - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y)\xi - 2n(f_1 - f_3)(\eta(Y)X - \eta(X)Y)$$  \hspace{1cm} (37)$$

Putting $Y = \xi$ in above equation, we get

$$QX = 2n(f_1 - f_3)(2\eta(X)\xi - X)$$  \hspace{1cm} (38)$$

Now, taking the inner product of the above equation with $U$, we get

$$S(X,U) = 2n(f_1 - f_3)\{ g(X,U) - 2\eta(X)\eta(U) \}$$  \hspace{1cm} (39)$$

which shows that generalised Sasakian-space-form is an $\eta$– Einstein Manifold. Conversely, suppose that (39) is satisfied. Then by virtue of (35) and (38), we get $I(X,Y)\xi = 0$. \hfill $\square$

5. Generalized Sasakian-Space-Form Satisfying $I.S = 0$

Theorem 5.1. A $(2n+1)$– dimensional $(n > 1)$ generalised Sasakian-space-form satisfying $I.S = 0$ is an $\eta$– Einstein Manifold.

Proof. Let us consider generalised Sasakian-space-form $M^{2n+1}$ satisfying $I(\xi,X)Y, Z) + S(Y, I(\xi,X)Z) = 0$ for any vector fields $X, Y, Z$ on $M$. Substituting (21) in above equation, we obtain

$$2g(X,Y)S(\xi, Z) - \eta(X)S(Y, Z) + \eta(Y)S(X, Z) + 2S(Y, \xi)g(X,Z) - \eta(X)S(Y, Z) - \eta(Z)S(Y, X) = 0$$  \hspace{1cm} (40)$$

For $Z = \xi$, the last equation is equivalent to

$$2.2n(f_1 - f_3)g(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) - S(Y, X) = 0$$  \hspace{1cm} (41)$$

which implies that,

$$S(X,Y) = 2n(f_1 - f_3)\{ 2g(X,Y) - \eta(X)\eta(Y) \}$$  \hspace{1cm} (42)$$

This proves our assertion. \hfill $\square$

6. $W_9$– flat Generalized Sasakian-space-forms

Theorem 6.1. A $(2n+1)$– dimensional $(n > 1)$ generalised Sasakian-space-form is $W_9$– flat if and only if $f_1 = \frac{3f_2}{1-2n} = f_3$.

Proof. For a $(2n + 1)$– dimensional $(n > 1)$ $W_9$– flat generalised Sasakian-space-forms, we have from (18)

$$R(X,Y)Z = \frac{1}{2n} \{ S(X,Y)Z - g(Y,Z)QX \}$$  \hspace{1cm} (43)$$
In view of (9) and (10), the above equation takes the form

\[ R(X, Y)Z = \frac{1}{2n} \left\{ (2nf_1 + 3f_2 - f_3)(g(X, Y)Z - g(Y, Z)X) - (3f_2 + (2n - 1)f_3)(\eta(X)\eta(Y)Z + g(Y, Z)\eta(X)) \right\} \]  

(44)

By virtue of (8) the above equation reduces to

\[
\begin{align*}
&f_1 [g(Y, \phi Z)X - g(X, \phi Z)Y] + f_2 [g(X, \phi^2 Z)\phi Y - g(Y, \phi^2 Z)\phi X + 2g(X, \phi Y)\phi Z] \\
&+ f_3 [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi] \\
&= \frac{1}{2n} \left\{ (2nf_1 + 3f_2 - f_3)(g(X, Y)Z - g(Y, Z)X) - (3f_2 + (2n - 1)f_3)(\eta(X)\eta(Y)Z + g(Y, Z)\eta(X)) \right\}
\end{align*}
\]

(45)

Now, replacing \( Z \) by \( \phi Z \) in the above equation, we obtain

\[
\begin{align*}
f_1 [g(Y, \phi Z)X - g(X, \phi Z)Y] + f_2 [g(X, \phi^2 Z)\phi Y - g(Y, \phi^2 Z)\phi X + 2g(X, \phi Y)\phi Z] + f_3 [g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi] \\
= \frac{1}{2n} \left\{ (2nf_1 + 3f_2 - f_3)(g(X, Y)Z - g(Y, Z)X) - (3f_2 + (2n - 1)f_3)(\eta(X)\eta(Y)\phi Z + g(Y, \phi Z)\eta(X)) \right\}
\end{align*}
\]

(46)

Taking inner product of above equation with \( \xi \), we get

\[
\begin{align*}
f_1 [g(Y, \phi Z)\eta(X) - g(X, \phi Z)\eta(Y)] + f_2 [g(X, \phi Z)\eta(Y) - g(Y, \phi Z)\eta(X)] \\
= \frac{1}{2n} \left\{ (2nf_1 + 3f_2 - f_3)(-g(Y, \phi Z)\eta(X)) - (3f_2 + (2n - 1)f_3)g(Y, \phi Z)\eta(X) \right\}
\end{align*}
\]

(47)

Putting \( X = \xi \) in above equation, we get

\[ (4nf_1 + 6f_2 - 2f_3)g(Y, \phi Z) = 0 \]

(48)

Since \( g(Y, \phi Z) \neq 0 \) in general, we obtain

\[ 4nf_1 + 6f_2 - 2f_3 = 0 \]

(49)

Again replacing \( X \) by \( \phi X \) in equation (45), we get

\[
\begin{align*}
f_1 [g(Y, Z)\phi X - g(\phi X, Z)Y] + f_2 [g(\phi X, Z)\phi Y - g(Y, \phi Z)\phi X + 2g(\phi X, \phi Y)\phi Z] + f_3 [\eta(Y)\eta(Z)\phi X + g(\phi X, Z)\eta(Y)\xi] \\
= \frac{1}{2n} \left\{ (2nf_1 + 3f_2 - f_3)(g(\phi X, Y)Z - g(Y, Z)\phi X) \right\}
\end{align*}
\]

(50)

Taking inner product with \( \xi \)

\[
\begin{align*}
f_1 [\eta(Y)\eta(Z)\phi X + g(\phi X, Z)\eta(Y)] + f_3 g(\phi X, Z)\eta(Y) = \frac{1}{2n} (2nf_1 + 3f_2 - f_3)g(\phi X, Y)\eta(Z)
\end{align*}
\]

(51)

Putting \( Y = \xi \), we get

\[ (f_1 - f_3)g(\Phi X, Z) = 0 \]

(52)

Since \( g(\phi X, Z) \neq 0 \) in general, we obtain

\[ f_3 = f_1 \]

(53)

From equation (49) and (53), we get

\[ f_1 = \frac{3f_2}{1 - 2n} = f_3 \]
Conversely, suppose that \( f_1 = \frac{3f_2}{1-2n} = f_3 \) satisfies in generalized Sasakian-space-form and then we have

\[
S(X, Y) = 0, \quad (55)
\]
\[
QX = 0 \quad (56)
\]

Also, in view of (18), we have

\[
I(X, Y, Z, U) = 'R(X, Y, Z, U) \quad (57)
\]

where \( I(X, Y, Z, U) = g(X, Y, Z, U) \) and \( 'R(X, Y, Z, U) = g(X, Y, Z, U) \). Putting \( Y = Z = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get

\[
2n f_1 g(X, U) + 3f_2 g(\phi X, \phi U) - f_3 \{(2n + 1)\eta(X)\eta(U) + g(X, U)\} = 0 \quad (61)
\]

Putting \( X = U = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get \( f_1 = 0 \). Then in view of (54), \( f_2 = f_3 = 0 \). Therefore, we obtain from (8)

\[
R(X, Y)Z = 0 \quad (62)
\]

Hence in view of (55), (56) and (62), we have \( I(X, Y)Z = 0 \). This completes the proof.

### 7. Generalized Sasakian-space-forms Satisfying \( I.R = 0 \)

**Theorem 7.1.** A generalized Sasakian-space-form \( M^{2n+1}(f_1, f_2, f_3) \) satisfies the condition \( I(\xi, X).R = 0 \) if and only if the functions \( f_1 \) and \( f_3 \) has the sectional curvature \( (f_1 - f_3) \).

**Proof.** Let generalized Sasakian-space-form satisfying

\[
I(\xi, X)R(Y, Z)U = 0 \quad (63)
\]

This can be written as

\[
I(\xi, X)R(Y, Z)U - R(I(\xi, X)Y, Z)U - R(Y, I(\xi, X)Z)U - R(Y, Z)I(\xi, X)U = 0 \quad (64)
\]
for any vector fields $X, Y, Z, U$ on $M$. In view of (21), we obtain

$$I(\xi, X)R(Y, Z)U = (f_1 - f_3)\{2g(X, R(Y, Z)U)\xi - \eta(X)R(Y, Z)U - \eta(R(Y, Z)U)X\}$$  \hspace{1cm} (65)

On the other hand, by direct calculations, we have

$$R(I(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)R(\xi, Z)U - \eta(\xi)R(Y, Z)U - \eta(Y)R(\xi, Z)U\}$$ \hspace{1cm} (66)

$$R(Y, I(\xi, X)Z)U = (f_1 - f_3)\{2g(X, Z)R(\xi, Y)U - \eta(\xi)R(Y, Z)U - \eta(Z)R(Y, X)U\}$$ \hspace{1cm} (67)

$$R(Y, Z)I(\xi, X)U = (f_1 - f_3)\{2g(X, U)R(Y, Z)\xi - \eta(X)R(Y, Z)U - \eta(U)R(Y, Z)X\}$$ \hspace{1cm} (68)

Substituting (64), (65), (66) and (67) in (63), we get

$$(f_1 - f_3)\{2g(X, R(Y, Z)U)\xi - \eta(X)R(Y, Z)U - \eta(R(Y, Z)U)X - 2g(X, Y)R(\xi, Z)U + \eta(X)R(Y, Z)U + \eta(Y)R(\xi, Z)U$$
$$- 2g(X, Z)R(Y, X)U + \eta(Z)R(Y, Z)U - 2g(X, U)R(Y, Z)\xi + \eta(X)R(Y, Z)U + \eta(U)R(Y, Z)X = 0$$  \hspace{1cm} (69)

Taking inner product with $\xi$, above equation implies that

$$(f_1 - f_3)\{2g(X, R(Y, Z)U) - \eta(X)\eta(R(Y, Z)U) - 2g(X, Y)\eta(R(\xi, Z)U) + \eta(Y)\eta(R(X, Z)U) - 2g(X, Z)\eta(R(\xi, Y)U)$$
$$+ 2\eta(X)\eta(R(Y, Z)U) + \eta(Z)\eta(R(Y, X)U) - 2g(X, U)\eta(R(Y, Z)\xi) + \eta(U)\eta(R(Y, Z)X) = 0$$ \hspace{1cm} (70)

In consequence of (8), (12), (13) and (14) the above equation takes the form

$$2g(X, R(Y, Z)U) - 2(f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U)) + (f_1 - f_3)\{g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U)\} = 0$$

On solving, we get $2g(X, R(Y, Z)U) - (f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U) = 0$, which say us $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature $(f_1 - f_3)$.

\section*{8. Generalized Sasakian-space-forms satisfying $I.P = 0$}

\textbf{Theorem 8.1.} A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $I(\xi, X)P = 0$ if and only if $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature of the form $(f_1 - f_3)$.

\textbf{Proof.} The condition $I(\xi, X)P = 0$ implies that

$$(I(\xi, X)P)(Y, Z, U) = I(\xi, X)P(Y, Z)U - P(I(\xi, X)Y, Z)U - P(Y, I(\xi, X)Z)U - P(Y, Z)I(\xi, X)U = 0$$ \hspace{1cm} (71)

for any vector fields $X, Y, Z$ on $M$. In view of (10), we obtain from (27)

$$\eta(P(X, Y)Z) = 0$$ \hspace{1cm} (72)

Since,

$$I(\xi, X)P(Y, Z)U = (f_1 - f_3)\{2g(X, P(Y, Z)U)\xi - \eta(X)P(Y, Z)U\}$$ \hspace{1cm} (73)

$$P(I(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)P(\xi, Z)U - \eta(\xi)P(Y, Z)U - \eta(Y)P(X, Z)U\}$$ \hspace{1cm} (74)

$$P(Y, I(\xi, X)Z)U = (f_1 - f_3)\{2g(X, Z)P(\xi, Y)U - \eta(X)P(Y, Z)U - \eta(Z)P(Y, X)U\}$$ \hspace{1cm} (75)
Finally, we conclude that
\[ P(Y, Z)I(\xi, X)U = (f_1 - f_3)\{2g(X, U)P(Y, Z)\xi - \eta(X)P(Y, Z)U - \eta(U)P(Y, Z)X\} \]  
(76)

So, substituting (73), (74), (75) and (76) in (63), we deduce that
\[ (f_1 - f_3)\{2g(X, P(Y, Z)U)\xi - \eta(X)P(Y, Z)U - 2g(X, Y)P(\xi, Z)U + \eta(X)P(Y, Z)U + \eta(Y)P(X, Z)U - 2g(X, Z)P(Y, \xi)U + \eta(Y)P(Y, Z)U + \eta(Z)P(Y, X)U - 2g(X, U)P(Y, Z)\xi + \eta(X)P(Y, Z)U + \eta(U)P(Y, Z)X\} = 0 \]
(77)

Taking inner product with \( \xi \), we get
\[ (f_1 - f_3)\{g(X, R(Y, Z)U) - (f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U))\} = 0 \]
which say us \( M^{2n+1}(f_1, f_2, f_3) \) has the sectional curvature \( (f_1 - f_3) \). \( \square \)

9. Generalized Sasakian-space-forms Satisfying \( I.\tilde{C} = 0 \)

**Theorem 9.1.** A generalized Sasakian-space-forms \( M^{2n+1}(f_1, f_2, f_3) \) satisfies the condition \( I(\xi, X)\tilde{C} = 0 \) if and only if either the scalar curvature \( \tau \) of \( M^{2n+1}(f_1, f_2, f_3) \) is \( \tau = 8n(2n + 1)(f_1 - f_3) \) or a real space form with the sectional curvature \( (f_1 - f_3) \).

**Proof.** The condition \( I(\xi, X)\tilde{C} = 0 \) implies that
\[ I(\xi, X)\tilde{C}(Y, Z)U = I(\xi, X)\tilde{C}(Y, Z)U - \tilde{C}(I(\xi, X)Y, Z)U - \tilde{C}(I(\xi, X)Z, Y)U - \tilde{C}(Y, I(\xi, X)Z)U = 0 \]
(78)

for any vector fields \( X, Y, Z \) and \( U \) on \( M \). From (22) and (23), we can easily to see that
\[ I(\xi, X)\tilde{C}(Y, Z)U = (f_1 - f_3)\{2g(X, \tilde{C}(Y, Z)U)\xi - \eta(X)\tilde{C}(Y, Z)U - \eta(\tilde{C}(Y, Z)U)X\} \]
(79)
\[ \tilde{C}(I(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)\tilde{C}(\xi, Z)U - \eta(X)\tilde{C}(Y, Z)U - \eta(Y)\tilde{C}(X, Z)U\} \]
(80)
\[ \tilde{C}(Y, I(\xi, X)Z)U = (f_1 - f_3)\{2g(X, Z)\tilde{C}(Y, \xi)U - \eta(X)\tilde{C}(Y, Z)U - \eta(Z)\tilde{C}(Y, X)U\} \]
(81)

and
\[ \tilde{C}(Y, Z)I(\xi, X)U = (f_1 - f_3)\{2g(X, U)\tilde{C}(Y, Z)\xi - \eta(X)\tilde{C}(Y, Z)U - \eta(U)\tilde{C}(Y, Z)X\} \]
(82)

Thus, substituting (79), (80), (81) and (82) in (78) and after from necessary abbreviations, (78) takes from
\[ (f_1 - f_3)\{2g(X, \tilde{C}(Y, Z)U)\xi - \eta(X)\tilde{C}(Y, Z)U - \eta(\tilde{C}(Y, Z)U)X - 2g(X, Y)\tilde{C}(\xi, Z)U + \eta(X)\tilde{C}(Y, Z)U + \eta(Y)\tilde{C}(X, Z)U - 2g(X, Z)\tilde{C}(Y, \xi)U - \eta(X)\tilde{C}(Y, Z)U - \eta(Z)\tilde{C}(Y, X)U - 2g(X, U)\tilde{C}(Y, Z)\xi + \eta(X)\tilde{C}(Y, Z)U + \eta(U)\tilde{C}(Y, Z)X\} = 0 \]
(83)

Taking inner product with \( \xi \) and solving
\[ (f_1 - f_3)\{2g(X, R(Y, Z)U) + (f_1 - f_3)(g(Z, U)g(Y, X) - g(Y, U)g(X, Z)) + \left(f_1 - f_3 - \frac{\tau}{2n(2n + 1)}\right)(2g(Z, U)\eta(X)\eta(Y) - 2g(Y, U)\eta(X)\eta(Z) + g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U))\} = 0 \]
(84)
Now putting $U = \xi$ in the above equation, we get

$$(f_1 - f_3)(4(f_1 - f_3) - \frac{\tau}{2n(2n+1)})\{g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\} = 0$$

Above equation tells us that $M^{2n+1}(f_1, f_2, f_3)$ has the scalar curvature $\tau = 8n(2n + 1)(f_1 - f_3)$. Conversely, if $M^{2n+1}(f_1, f_2, f_3)$ is either real space form with sectional curvature $(f_1 - f_3)$ or it has the scalar curvature $\tau = 8n(2n + 1)(f_1 - f_3)$. This completes the proof.

References


New Separation Axioms in Soft Bitopological Space

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Abstract: The present paper introduces a new class of separation axioms called \((1,2)^\ast\)-soft b-separation axioms using \((1,2)^\ast\)-soft b-open set. Also the properties of \((1,2)^\ast\)-soft b\(T_i\)-spaces \((i = 0,1,2)\) are soft bitopological properties under the bijection and irresolute open soft mapping. Further, we show that the properties of \((1,2)^\ast\)-soft b\(T_i\)-spaces \((i = 0,1,2)\) are hereditary properties.

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1. Introduction

In real life situation, the problems in economics, engineering, social sciences, medical sciences etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of these theories were given like theory of fuzzy sets, rough set which we can use as mathematical tools for dealings with uncertainties. But all these theories have their own difficulties. The reason for these difficulties Molodtsov [6] initiated the concept of soft set theory as a new mathematical tools for dealings with uncertainties which is free from the above difficulties. Molodtsov successfully applied soft set theory in several direction, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement and so on.

In 1963, J.C. Kelly [5], first initiated the concept of bitopological spaces. After then many authors studied some of basic concepts and properties of bitopological space. In 1996, Andrijevic [1] introduced a new class of open sets in a topological space called b-open sets. Recently, in 2011, Shabir and Naz [7] initiated the study of the soft topological spaces. They defined soft topology as a collection of soft sets over \(X\). Also they defined basic notations of soft topological spaces such as soft open and soft closed sets, soft subspace, soft closure, soft interior, soft separation axioms and established their several properties. Metin Akdag and Alkan Ozkan [11] are defined soft b-open sets and soft b-continuous map studied their properties. In the year 2014, Basavaraj M.Ittanagi [2] initiated the concept of soft bitopological spaces which are defined over an initial universe with a fixed set of parameters.

In the present paper, we introduce a new class of separation axioms called \((1,2)^\ast\)-soft b-separation axioms using \((1,2)^\ast\)-soft b-open set. In particular we study the properties of the \((1,2)^\ast\)-soft b\(T_0\) spaces, \((1,2)^\ast\)-soft b\(T_1\)-spaces and \((1,2)^\ast\)-soft b-Haustroff spaces. we give the characterizations of these spaces.
2. Preliminaries

Throughout this paper, X is an initial universe, E is the set of parameters, P(X) is the power set of X and A is a nonempty subset of E.

**Definition 2.1** ([7]). A soft set $F_A$ on the universe $X$ is defined by the set of ordered pairs $F_A = \{(x, f_A(x)) : x \in E\}$, where $f_A : E \rightarrow P(X)$ such that $f_A(x) = \phi$ if $x \in A$. Here $f_A$ is called approximate function of the soft set $F_A$. The value of $f_A(x)$ may be arbitrary, some of them may be empty, some may have non empty intersection. The set of all soft sets over $X$ will be denoted by $S(X)$.

**Definition 2.2** ([7]). For two soft sets $F_A$, $G_B$ over a common universe $X$, we say that $F_A$ is a soft subset of $G_B$ if

1. $A \subseteq B$ and
2. For all $e \in A$, $F(e)$ and $G(e)$ are identical approximations

We write $F_A \subseteq G_B$. $F_A$ is said to be a soft super set of $G_B$ if $G_B$ is a soft subset of $F_A$. We denoted it by $F_A \supseteq G_B$.

**Definition 2.3** ([7]). Two soft sets $F_A$ and $G_B$ over the common universe $X$ are said to be soft equal if $F_A$ is a soft subset of $G_B$ and $G_B$ is a soft subset of $F_A$.

**Definition 2.4** ([7]). The soft union of two soft sets of $F_A$ and $G_B$ over the common universe $X$ is the soft set $H_C$, where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A \setminus B \\ G(e), & \text{if } e \in B \setminus A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

**Definition 2.5** ([7]). The soft intersection $H_C$ of two soft sets $F_A$ and $G_B$ over a common universe $X$, denoted by $F_A \cap G_B$, is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$, for all $e \in C$.

**Definition 2.6** ([7]). A soft set $F_E$ over $X$ is said to be a null soft set or empty soft set denoted by $\phi$ if for all $e \in E$, $F(e) = \phi$. It means that there is no element in $X$ related to the parameter $e \in E$. Therefore, we can’t display such elements in the soft sets, as it is meaningless to consider such parameters.

**Definition 2.7** ([7]). A soft set $F_E$ over $X$ is said to be an absolute soft set denoted by $\tilde{X}$ or $F_E$ if for all $e \in E$, $F(e) = X$. Clearly $\tilde{X}^C = \phi$ and $\phi^C = \tilde{X}$.

**Definition 2.8** ([7]). Let $F_E$ be a soft set over $X$ and $Y$ be a non empty subset of $X$. Then the soft set of $F_E$ over $Y$ denoted by $(Y : F_E)$ is defined as follows: $(Y : F_E) = Y \cap F(\alpha)$, for all $\alpha \in Y$. In other words, $(Y : F_E) = Y \cap I_{F_E}$.

**Definition 2.9** ([4]). Let $F_E \subseteq S(X)$. We say that $x_e = (e, \{x\})$ is a soft point of $F_E$ if $e \in E$ and $x \in F(e)$.

**Definition 2.10** ([4]). The soft point $x_e$ said to be belonging to the soft set $F_E$, denoted by $x_e \subseteq F_E$.

**Definition 2.11** ([3]). Let $F_A \subseteq S(X)$. The soft power set of $F_A$ is defined by $\tilde{P}(A) = \{F_A : F_A \subseteq F_A, i \in I \subseteq N\}$ and its cardinality is defined by $|\tilde{P}(A)| = 2^{|\tilde{I} \times A(i)|}$, where $|f_A(x)|$ is the cardinality of $f_A(x)$.

**Example 2.12.** [3] Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ then $\tilde{X} = \{(e_1, \{x_1, x_2\}), (e_2, \{x_1, x_2\})\}$. The possible soft subsets are $F_{E_1} = \{(e_1, \{x_1\}), (e_2, \{x_2\})\}$, $F_{E_2} = \{(e_1, \{x_2\})\}$, $F_{E_3} = \{(e_1, \{x_1, x_2\})\}$, $F_{E_4} = \{(e_2, \{x_1\})\}$, $F_{E_5} = \{(e_2, \{x_2\})\}$, $F_{E_6} = \{(e_2, \{x_1, x_2\})\}$, $F_{E_7} = \{(e_1, \{x_1\}), (e_2, \{x_1\})\}$, $F_{E_8} = \{(e_1, \{x_2\}),(e_2, \{x_2\})\}$, $F_{E_9} = \{(e_1, \{x_1\}),(e_2, \{x_1, x_2\})\}$, $F_{E_{10}} = \{(e_1, \{x_2\}),(e_2, \{x_1\})\}$, $F_{E_{11}} = \{(e_1, \{x_2\}),(e_2, \{x_2\})\}$, $F_{E_{12}} = \{(e_1, \{x_2\}),(e_2, \{x_1, x_2\})\}$, $F_{E_{13}} = \{(e_1, \{x_2\}),(e_2, \{x_2\})\}$, $F_{E_{14}} = \{(e_1, \{x_1, x_2\}),(e_2, \{x_2\})\}$, $F_{E_{15}} = \phi$, $F_{E_{16}} = \tilde{X}$. 

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Definition 2.13 ([7]). Let \( \tilde{\tau} \) be the collection of soft sets over \( X \), then \( \tilde{\tau} \) is said to be a Soft Topology on \( \tilde{X} \) if

1. \( \phi, \tilde{X} \) belongs to \( \tilde{\tau} \).

2. The soft union of any number of soft sets in \( \tilde{\tau} \) belongs to \( \tilde{\tau} \).

3. The soft intersection of any two soft sets in \( \tilde{\tau} \) belongs to \( \tilde{\tau} \).

The triplet \((\tilde{X}, \tilde{\tau}, E)\) is called a Soft Topological Space over \( X \).

Definition 2.14 ([11]). Let \( \tilde{X} \) be a non-empty soft set on the universe \( X \) with a parameter set \( E \) and \( \tilde{\tau}_1, \tilde{\tau}_2 \) are two different soft topologies on \( \tilde{X} \). Then \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) is called a soft bitopological space.

Definition 2.15 ([11]). Let \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) be a soft bitopological space and \( F_A \subseteq \tilde{X} \). Then \( F_A \) is called \( \tilde{\tau}_1\)-open if \( F_A = F_B \cup F_C \), where \( F_B \subseteq \tilde{\tau}_1 \) and \( F_C \subseteq \tilde{\tau}_2 \). The soft complement of \( \tilde{\tau}_1\)-open set is called \( \tilde{\tau}_1\)-closed.

Definition 2.16 ([9]). Let \( \tilde{X} \) be a soft bitopological space and \( F_A \subseteq \tilde{X} \). Then \( F_A \) is called \((1,2)^*\)-soft b-open set (briefly \((1,2)^*\)-sb-open) if \( F_A \subseteq \tilde{\tau}_1\text{-int}(\tilde{\tau}_2\text{-cl}(F_A)) \cup \tilde{\tau}_1\text{-cl}(\tilde{\tau}_2\text{-int}(F_A)) \).

Definition 2.17 ([9]). Let \( \tilde{X} \) be a soft bitopological space and \( F_A \) be a soft set over \( \tilde{X} \).

1. \((1,2)^*\)-soft b-closure (briefly \((1,2)^*\)-sbcl\((F_A)\)) of a set \( F_A \) in \( \tilde{X} \) is defined by \((1,2)^*\)-sbcl\((F_A)\) = \( \bigcap \{ F_B \subseteq F_A : F_E \text{ is a (1,2)*-soft b-closed set in } \tilde{X} \} \).

2. \((1,2)^*\)-soft b-interior (briefly \((1,2)^*\)-sbi\((F_A)\)) of a set \( F_A \) in \( \tilde{X} \) is defined by \((1,2)^*\)-sbi\((F_A)\) = \( \bigcup \{ F_B \subseteq F_A : F_E \text{ is a (1,2)*-soft b-open set in } \tilde{X} \} \).

Definition 2.18 ([7]). Let \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) be a soft bitopological space over \( X \) and \( Y \) be non empty subset of \( X \). Then \( \tilde{\tau}_1Y = \{(Y,F_E) : F_E \subseteq \tilde{\tau}_1 \} \) and \( \tilde{\tau}_2Y = \{(Y,F_E) : F_E \subseteq \tilde{\tau}_2 \} \) are said to be the relative soft topologies on \( \tilde{Y} \). Then \( \{\tilde{Y}, \tilde{\tau}_1Y, \tilde{\tau}_2Y, E\} \) is called the relative soft bitopological space of \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\).

Definition 2.19 ([10]). A soft mapping \( \tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E) \) is said to be \((1,2)^*\)-soft b-continuous (briefly \((1,2)^*\)-sb-continuous) if the inverse image of each \( \tilde{\tau}_1\)-open set of \( \tilde{Y} \) is \((1,2)^*\)-sb-open set in \( \tilde{X} \).

Definition 2.20 ([10]). A soft mapping \( \tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E) \) is said to be \((1,2)^*\)-soft b-irresolute (briefly \((1,2)^*\)-sb-irresolute) if \( \tilde{f}^{-1}(F_A) \) is a \((1,2)^*\)-sb-closed set in \( \tilde{X} \), for every \((1,2)^*\)-sb-closed set \( F_A \) in \( \tilde{Y} \).

Definition 2.21. Let \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) be a soft bitopological space over \( X \) and \( F_E \subseteq S(X) \). \( x_e \in \tilde{X} \) is said to be a \((1,2)^*\)-soft \( b \)-limit point of \( F_E \) if every \((1,2)^*\)-soft \( b \)-neighbourhood containing \( x_e \) contains a soft point of \( F_E \) other than \( x_e \).

Definition 2.22. The collection of all \((1,2)^*\)-soft \( b \)-limit points of \( F_E \) is called the \((1,2)^*\)-soft \( b \)-derived set of \( F_E \) and is denoted by \((1,2)^*\)-\( sbD(F_E) \).

Definition 2.23 ([10]). A soft mapping \( \tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E) \) is said to be \((1,2)^*\)-soft b-open map (briefly \((1,2)^*\)-sb-open) if the image of every \( \tilde{\tau}_1\)-open set of \( \tilde{X} \) is \((1,2)^*\)-sb-open set in \( \tilde{Y} \).

Definition 2.24 ([2]). Let \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) be a soft bitopological space over \( X \). Then the \((1,2)^*\)-soft \( T_i \) axioms (where \( i = 0, 1, 2 \)) are as follows.

\((1,2)^*\)-Soft \( T_0 \) axiom : If for every \( x_e, y_e \in \tilde{X} \) with \( x_e \neq y_e \), there exist \( \tilde{\tau}_1\)-open sets \( F_{E_1} \) and \( F_{E_2} \) such that either \( x_e \in F_{E_1} \) but \( y_e \notin F_{E_1} \) or \( y_e \in F_{E_2} \) but \( x_e \notin F_{E_2} \).
(1, 2)*-Soft $T_1$ axiom : If for every $x_e$, $y_e \in \tilde{X}$ with $x_e \neq y_e$, there exist $\tau_{1,2}$-open sets $F_{E_1}$ and $F_{E_2}$ such that $x_e \in F_{E_1}$ but $y_e \notin F_{E_1}$ and $y_e \in F_{E_2}$ but $x_e \notin F_{E_2}$.

(1, 2)*-Soft $T_2$ axiom : If for every $x_e$, $y_e \in \tilde{X}$ with $x_e \neq y_e$, there exist $\tau_{1,2}$-open sets $F_{E_1}$ and $F_{E_2}$ such that $x_e \in F_{E_1}$, $y_e \in F_{E_2}$ and $F_{E_1} \cap F_{E_2} = \phi$.

3. (1, 2)*-soft b-Separation Axioms

In this section, we introduce and study the new concepts of (1, 2)*-soft b-separation axioms and investigated basic properties of these concepts in soft bitopological spaces.

**Definition 3.1.** Let $(\tilde{X}, \tau_1, \tau_2)$ be a soft bitopological space over $X$ and for every soft points $x_e, y_e \in \tilde{X}$ with $x_e \neq y_e$. Then the soft bitopological space $(\tilde{X}, \tau_1, \tau_2)$ is said to be a (1, 2)*-soft bT$_0$-space ([1, 2)*-sbT$_0$-space) if there exists (1, 2)*-soft b-open sets $F_{E_1}$ and $F_{E_2}$ such that either $x_e \in F_{E_1}$ but $y_e \notin F_{E_1}$ or $y_e \in F_{E_2}$ but $x_e \notin F_{E_2}$.

**Example 3.2.** Let $X = \{x, y\}$, $E = \{e_1, e_2\}$, and $\tilde{X} = \{(e_1, \{x, y\}), (e_2, \{x, y\})\}$. The possible soft subsets are considered as in Example 2.12. Define $\tilde{\tau}_1 = (\tilde{X}, \tau_1, \tau_2)$ and $\tilde{\tau}_2 = (\tilde{X}, \tau_1, \tau_2)$. Then $\tilde{\tau}_{1,2}$-open sets are $(\tilde{X}, F_{E_1}, F_{E_2}, F_{E_3}, F_{E_4})$ and the collection of all (1, 2)*-soft b-open set is $(1, 2)*$-sbO$(\tilde{X}) = \{\tilde{X}, \phi, F_{E_1}, F_{E_2}, F_{E_3}, F_{E_4}, F_{E_5}, F_{E_6}, F_{E_7}, F_{E_8}\}$. Then $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a (1, 2)*-soft bT$_0$-space over $X$.

**Remark 3.1.** Every (1, 2)*-soft bT$_0$-space is soft bitopological space. But the following example shows that every soft bitopological space need not be (1, 2)*-soft bT$_0$-space.

**Example 3.3.** Consider the soft indiscrete bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2)$ over $X$. The only (1, 2)*-soft b-open sets are $\phi$ and $\tilde{X}$. Now, the (1, 2)*-soft b-open set $\tilde{X}$ contains $x_e$, but it also contains $y_e$. Thus, there is no (1, 2)*-soft b-open set which contains $x_e$ but does not contain $y_e$. Hence, it is not a (1, 2)*-soft bT$_0$-space.

**Proposition 3.4.** Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2)$ be a soft bitopological space over $X$ and $x_e, y_e \in \tilde{X}$ with $x_e \neq y_e$, then there exists (1, 2)*-soft b-open sets $F_{E_1}$ and $F_{E_2}$, such that either $x_e \in F_{E_1}$ and $y_e \notin F_{E_1}$ or $y_e \in F_{E_2}$ and $x_e \notin F_{E_2}$. Then, the soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a (1, 2)*-soft bT$_0$-space.

**Proof.** Let $x_e, y_e \in \tilde{X}$ with $x_e \neq y_e$. Let $F_{E_1}$ and $F_{E_2}$ be (1, 2)*-soft b-open sets such that either $x_e \in F_{E_1}$ and $y_e \notin F_{E_1}$ or $y_e \in F_{E_2}$ and $x_e \notin F_{E_2}$. Then $x_e \in F_{E_1}$ and $y_e \notin F_{E_1}$, then $y_e \in (F(e))^{\tilde{\tau}_1}$ for all $e \in E$. Therefore $y_e \notin F_{E_1}$. Similarly, if $y_e \in F_{E_2}$ and $x_e \notin F_{E_2}$. Hence $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a (1, 2)*-soft bT$_0$-space.

A characterization for (1, 2)*-soft bT$_0$-space is following.

**Theorem 3.5.** A soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is (1, 2)*-soft bT$_0$-space over $X$ if and only if $(1, 2)*$-sbcl $\{x_e\} \neq (1, 2)*$-sbcl$\{y_e\}$ for every pair of distinct soft point $x_e$, $y_e$ of $\tilde{X}$.

**Proof.** Let $x_e, y_e \in \tilde{X}$ with $x_e \neq y_e$. Since $\tilde{X}$ is (1, 2)*-soft bT$_0$-space, then there exists (1, 2)*-soft b-open sets $F_{E_1}$ and $G_{E_1}$ such that either $x_e \notin F_{E_1}$ but $y_e \in F_{E_1}$ or $y_e \in G_{E_1}$ but $x_e \notin G_{E_1}$. Since $\tilde{X} \setminus F_{E_1}$ is a (1, 2)*-soft b-closed set which does not contain $x_e$ but $y_e$. By definition, (1, 2)*-sbcl$\{y_e\}$ is the intersection of all (1, 2)*-soft b-closed set containing $y_e$. Therefore (1, 2)*-sbcl$\{y_e\}$ is a (1, 2)*-soft b-closed set. Hence $x_e \notin (1, 2)*$-sbcl$\{x_e\}$ but $x_e \in (1, 2)*$-sbcl$\{x_e\}$.

Conversely, assume that $x_e, y_e \in \tilde{X}$ with $x_e \neq y_e$ and $(1, 2)*$-sbcl $\{x_e\} \neq (1, 2)*$-sbcl$\{y_e\}$. Then by assumption, there exists at least one soft point $z_e \in \tilde{X}$ such that $z_e \in (1, 2)*$-sbcl$\{x_e\}$ but $z_e \notin (1, 2)*$-sbcl$\{y_e\}$.

Suppose not, $x_e \in (1, 2)*$-sbcl$\{y_e\}$ then $x_e \notin (1, 2)*$-sbcl$\{y_e\}$ which implies that $(1, 2)*$-sbcl$\{x_e\} \neq (1, 2)*$-sbcl$\{y_e\}$. Therefore $x_e \notin (1, 2)*$-sbcl$\{y_e\}$. Now we claim that $x_e \notin (1, 2)*$-sbcl$\{y_e\}$. Therefore $x_e \notin (1, 2)*$-sbcl$\{y_e\}$. Now we claim that $x_e \notin (1, 2)*$-sbcl$\{y_e\}$.
Theorem 3.6. A soft subspace of a $(1,2)^*\text{-}sbT_0$-space is a $(1,2)^*\text{-}sbT_0$-space.

Proof. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*\text{-}sbT_0$-space over $X$ and $(\tilde{Y}, \tilde{\tau}_1Y, \tilde{\tau}_2Y, E)$ be soft subspace of $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ over $Y$. Let $x_\in X_\in Y$ such that $x_\notin y_\in X$ and since $\tilde{Y}_\in \tilde{X}_\in X_\in Y$. Since $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-}sbT_0$-space over $X$, there exists $(1,2)^*\text{-}sbT_0$-open sets $F_1$ and $F_2$ such that $x_\in F_1$ but $y_\notin F_1$ or $y_\in F_2$ but $x_\notin F_2$. Now $x_\in \tilde{Y}_\in \tilde{F}_1 = \tilde{Y}_E$ which is a $(1,2)^*\text{-}sbT_0$-open set in $(\tilde{Y}, \tilde{\tau}_1Y, \tilde{\tau}_2Y, E)$. Consider $y_\notin F_1$, this implies that $y_\notin F(e)$ for some $e_\in E$. Therefore $y_\notin \tilde{Y}_\in \tilde{F}_1 = \tilde{Y}_E$. Similarly if $y_\notin F_2$ and $x_\notin F_2$, then $y_\notin \tilde{Y}_E$ and $x_\notin \tilde{Y}_E$. Thus $(\tilde{Y}, \tilde{\tau}_1Y, \tilde{\tau}_2Y, E)$ is also a $(1,2)^*\text{-}sbT_0$-space.

Theorem 3.7. Let $\tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \to (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ a bijective $(1,2)^*\text{-}sbT_0$-open mapping and if $\tilde{X}$ is a $(1,2)^*\text{-}sbT_0$-space, then $\tilde{Y}$ is a $(1,2)^*\text{-}sbT_0$-space.

Proof. Let $y_\in Y_\in X$ and $y_\in Y_\in X$. Since $\tilde{f}$ is bijective, there exists $x_\in X_\in Y$ such that $f(x_\in ) = y_\in X_\in Y$ and $\tilde{f}(x_\in ) = y_\in Y_\in X$. Since $\tilde{X}$ is a $(1,2)^*\text{-}sbT_0$-space, then there exists $\tilde{G}_1$ and $\tilde{G}_2$ such that $x_\in G_1$ but $x_\notin G_1$ or $x_\in G_2$ but $x_\notin G_2$. But $\tilde{f}$ is a $(1,2)^*\text{-}sbT_0$-open mapping, then $\tilde{f}(A_\in )$, $\tilde{f}(A_\in )$ are $(1,2)^*\text{-}sbT_0$-open sets in $\tilde{Y}$ with $y_\in \tilde{f}(A_\in )$ but $y_\notin \tilde{f}(A_\in )$ or $y_\in \tilde{f}(A_\in )$ but $y_\notin \tilde{f}(A_\in )$. Therefore $\tilde{Y}$ is a $(1,2)^*\text{-}sbT_0$-space.

Theorem 3.8. Let $\tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \to (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ a injective $(1,2)^*\text{-}sbT_0$-open b- irresolute mapping and if $\tilde{X}$ is a $(1,2)^*\text{-}sbT_0$-space, then $\tilde{X}$ is a $(1,2)^*\text{-}sbT_0$-space.

Proof. Let $x_\in X_\in Y_\in X$ with $x_\in Y_\in X$. Since $\tilde{f}$ is injective and $\tilde{Y}$ is a $(1,2)^*\text{-}sbT_0$-space, then there exists $(1,2)^*\text{-}sbT_0$-open sets $F_1$ and $F_2$ such that $x_\in F_1$ but $y_\notin F_1$ or $y_\in F_2$ but $x_\notin F_2$. Since $\tilde{f}$ is a $(1,2)^*\text{-}sbT_0$-open b- irresolute mapping, $\tilde{f}^{-1}(F_1)$ and $\tilde{f}^{-1}(F_2)$ are $(1,2)^*\text{-}sbT_0$-open sets in $\tilde{X}$ such that $x_\in \tilde{f}^{-1}(F_1)$ but $y_\notin \tilde{f}^{-1}(F_1)$ or $y_\in \tilde{f}^{-1}(F_2)$ but $x_\notin \tilde{f}^{-1}(F_2)$. Thus $\tilde{X}$ is a $(1,2)^*\text{-}sbT_0$-space.

Definition 3.9. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a soft bitopological space over $X$ and for every soft points $x_\in X_\in X$ with $x_\neq y_\in X$. Then the soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is said to be $(1,2)^*\text{-}sbT_1$-space if there exists $(1,2)^*\text{-}sbT_0$-open sets $F_1$ and $F_2$ such that $x_\in \tilde{F}_1$ but $y_\notin \tilde{F}_1$ and $y_\in \tilde{F}_2$ but $x_\notin \tilde{F}_2$.

Example 3.10. Let $X = \{x, y, z\}, E = \{e_1\}$ the soft subsets of $X$ is $SS_E(X)$ and $|S(X)| = 8$. They are $\tilde{X}$, $\phi$, $G_{e_1} = \{(e_1, \{x\})\}$, $G_{e_2} = \{(e_1, \{y\})\}$, $G_{e_3} = \{(e_1, \{z\})\}$, $G_{e_4} = \{(e_1, \{x, y\})\}$, $G_{e_5} = \{(e_1, \{x, z\})\}$, $G_{e_6} = \{(e_1, \{y, z\})\}$. Define $\tilde{\tau}_1 = \{\tilde{X}, \phi, G_{e_1}\}$ and $\tilde{\tau}_2 = \{\tilde{X}, \phi, G_{e_2}\}$. Then $\tilde{\tau}_1$-open sets are $\tilde{X}, \phi, G_{e_1}, G_{e_2}$. Then $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a soft bitopological space. The collection of $(1,2)^*\text{-}sbT_1$-open sets are $(1,2)^*\text{-}sbO(\tilde{X}) = \{\tilde{X}, \phi, G_{e_1}, G_{e_2}, G_{e_3}, G_{e_4}\}$ and $(1,2)^*\text{-}sbT_0$-closed sets are $(1,2)^*\text{-}sbC(\tilde{X}) = \{\tilde{X}, \phi, G_{e_1}, G_{e_2}, G_{e_3}, G_{e_4}\}$. Then this soft bitopological space is $(1,2)^*\text{-}sbT_1$-space.

Proposition 3.11. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a soft bitopological space over $X$ and $x_\in \in X_\in X$ such that $x_\neq y_\in X$. If there exists $(1,2)^*\text{-}sbT_0$-open sets $F_1$ and $F_2$ such that $x_\in F_1$ but $y_\notin F_1$ and $y_\in F_2$ but $x_\notin F_2$. Then the soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-}sbT_1$-space.

Proof. It is similar to the proof of proposition 3.4.

The following theorem is a characterization for $(1,2)^*\text{-}sbT_1$-space.
Theorem 3.12. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*\text{-soft bT}_1$-space over $X$ if and only if for each $x, y \in \tilde{X}$, every soft singleton$\{x\}$ over $X$ is $(1,2)^*\text{-soft b-closed set.}$

Proof. Suppose that $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-soft bT}_1$-space over $X$ and $x, y \in \tilde{X}$. Now we have to prove that the soft singleton set $\{x\}$ over $X$ is $(1,2)^*\text{-soft b-closed set.}$ Suppose $\{x\} \not\in (1,2)^*\text{-soft b-closed set.}$ Then $(1,2)^*\text{-soft b-cl}(\{x\}) \neq \{x\}$. So there exists $y, \neq x, y \in (1,2)^*\text{-soft b-cl}(\{x\})$. This contradicts the fact that $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-soft bT}_1$-space. Therefore, soft singleton$\{x\}$ over $X$ is $(1,2)^*\text{-soft b-closed set.}$

Conversely, suppose the soft singleton$\{x\}$ is $(1,2)^*\text{-soft b-closed for every } x \in \tilde{X}$. Since $\{x\} \not\in (1,2)^*\text{-soft b-closed,}$ $\{x\}^C$ is $(1,2)^*\text{-soft b-open set in } \tilde{X}$. Let $x, y \in \tilde{X}$ and $x \neq y$, such that $\{x\}$ and $\{y\}$ are $(1,2)^*\text{-soft b-closed sets,}$ then $\{x\}^C$ and $\{y\}^C$ are $(1,2)^*\text{-soft b-open sets.}$ Therefore $y \not\in \{x\}^C$ but $x \not\in \{y\}^C.$ Thus $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-soft bT}_1$-space over $X$.

Theorem 3.13. A subspace of $(1,2)^*\text{-soft bT}_1$-space is $(1,2)^*\text{-soft bT}_0$-space.

Proof. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*\text{-soft bT}_1$-space over $X$ and $(\tilde{Y}, \tilde{\tau}_{11}, \tilde{\tau}_{12}, E)$ be soft subspace of $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ over $Y$. Let $x, y \in \tilde{Y}$ such that $x \neq y$. Since $\tilde{Y} \subseteq \tilde{X}$, $y \in \tilde{X}$ and $x \neq y$. Since $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-soft bT}_1$-space over $X$, there exists $(1,2)^*\text{-soft b-open sets } E_1$ and $E_2$ in $\tilde{X}$ such that $x \in E_1$ but $y \not\in E_1$ and $y \in E_2$ but $x \not\in E_2$. Hence $y \in \tilde{Y} \cap E_1 \cap E_2 = \tilde{Y} \cap E_1 \cap E_2 = \tilde{Y} \cap E_1 \cap E_2$. Similarly if $y \in E_2$ and $x \not\in E_2$, then $y \in \tilde{Y} \cap E_2 \cap E_2 = \tilde{Y} \cap E_2 \cap E_2$. Thus $(\tilde{Y}, \tilde{\tau}_{11}, \tilde{\tau}_{12}, E)$ is also a $(1,2)^*\text{-soft bT}_0$-space.

Proposition 3.14. Every $(1,2)^*\text{-soft bT}_1$-space is $(1,2)^*\text{-soft bT}_0$-space.

Proof. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*\text{-soft bT}_1$-space. Then for every $x, y \in \tilde{X}$ with $x \neq y$, there exist $(1,2)^*\text{-soft b-closed sets } E_1$ and $E_2$ such that $x \in E_1$ but $y \neq E_1$ and $y \in E_2$ but $x \neq E_2$.

The converse of the above proposition need not be true.

Example 3.15. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$, $\tilde{X} = \{(e_1, \{x, y\}),(e_2, \{x, y\})\}$. The possible soft subsets are considered as in Example 2.12.

Define $\tilde{\tau}_1 = \{\tilde{X}, \phi, E_1, F_{E_1}\}$ and $\tilde{\tau}_2 = \{\tilde{X}, \phi, F_{E_1}\}$. Then $\tilde{\tau}_{1,2}\text{-soft open sets are } (\tilde{X}, F_{E_1}, F_{E_2}, F_{E_{13}})$ and the collection of all $(1,2)^*\text{-soft b-open sets are }\{X, \tilde{\tau}_{1,2}\text{-soft b-closed set}\}.$ Then $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*\text{-soft bT}_1$-space over $X$, but not $(1,2)^*\text{-soft bT}_0$-space over $X$. Since the soft singleton set $F_{E_1}$ is not a $(1,2)^*\text{-soft b-closed set.}$

Theorem 3.16. If any finite soft subset of a soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is $(1,2)^*\text{-soft b-closed set,}$ then $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is $(1,2)^*\text{-soft bT}_1$-space.

Proof. Let $x, y \in \tilde{X}$ with $x \neq y$. Then by hypothesis, $\{x\}$ and $\{y\}$ are $(1,2)^*\text{-soft b-closed sets,}$ which implies that $\{x\}^C$ and $\{y\}^C$ are $(1,2)^*\text{-soft b-open sets,}$ such that $x \in \{y\}^C$ and $y \in \{x\}^C$. Therefore $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is $(1,2)^*\text{-soft bT}_1$-space.

Theorem 3.17. Let $\tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ a bijective $(1,2)^*\text{-soft b-open mapping and if } \tilde{X}$ is a $(1,2)^*\text{-soft bT}_1$-space, then $\tilde{Y}$ is a $(1,2)^*\text{-soft bT}_1$-space.
Proof. Let $y_1, y_2$ be two distinct soft points of $\tilde{Y}$. Since $\tilde{f}$ is bijective, there exists $x_1, x_2 \in \tilde{X}$ such that $\tilde{f}(x_1) = y_1$ and $\tilde{f}(x_2) = y_2$. Since $\tilde{X}$ is a $(1,2)^*$-soft $T_1$-space, then there exist $\tilde{G}_E^1$ and $\tilde{G}_E^2$ of $\tilde{X}$ such that $x_1 \notin \tilde{G}_E^1$, $x_2 \notin \tilde{G}_E^2$, $x_1 \notin \tilde{G}_E^1$, and $x_2 \notin \tilde{G}_E^2$. But $\tilde{f}$ is a $(1,2)^*$-soft b-open mapping, then $\tilde{f}(E_1)$, $\tilde{f}(E_2)$ are $(1,2)^*$-soft b-open sets in $\tilde{Y}$ with $y_1 \notin \tilde{f}(E_1)$ but $y_2 \notin \tilde{f}(E_2)$ and $y_1 \notin \tilde{f}(E_2)$ but $y_2 \notin \tilde{f}(E_2)$. Therefore $\tilde{Y}$ is a $(1,2)^*$-soft $bT_1$-space.

Theorem 3.18. Let $\tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\sigma}_1, \tilde{\sigma}_2, E)$ be a injective $(1,2)^*$-soft $b$- irresolute mapping and if $\tilde{Y}$ is a $(1,2)^*$-soft $bT_1$-space, then $\tilde{X}$ is a $(1,2)^*$-soft $bT_1$-space.

Proof. The proof of the theorem is similar to the Theorem 3.8.

Definition 3.19. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a soft bitopological space over $X$ and for every soft points $x_1, y_1 \in \tilde{X}$ with $x_1 \neq y_1$. Then the soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is said to be a $(1,2)^*$-soft $bT_2$-space ((1,2)$^*$-soft $bH$- space) if there exists $(1,2)^*$-soft b-open sets $F_E^1$ and $F_E^2$ such that $x_1 \in F_E^1, y_1 \in F_E^2$ and $F_E^1 \cap F_E^2 = \phi$.

Example 3.20. Consider a $(1,2)^*$-soft discrete bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$. Let $x_1, y_1$ be two distinct soft points of $\tilde{X}$. And $\{x_1\}, \{y_1\}$ be $(1,2)^*$-soft b-open sets of $x_1$ and $y_1$, respectively such that $\{x_1\} \cap \{y_1\} = \phi$. Hence $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*$-soft $bT_2$-space or $(1,2)^*$-soft b- $bH$- space.

Theorem 3.21. A soft subspace of a $(1,2)^*$-soft $bT_2$-space is $(1,2)^*$-soft $bT_2$-space.

Proof. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*$-soft $bT_2$-space over $X$ and $(\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be soft subspace of $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ over $Y$. Let $x_1, y_1 \in \tilde{X}$ such that $x_1 \neq y_1$. Then $x_1, y_1 \in \tilde{X}$ and $x_1 \neq y_1$. Since $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*$-soft $bT_2$-space over $X$, there exists $(1,2)^*$-soft b-open sets $F_E^1$ and $F_E^2$ in $\tilde{X}$ such that $x_1 \in F_E^1$ and $y_1 \in F_E^2$ and $F_E^1 \cap F_E^2 = \phi$. It follows that $x_1 \in F_E^1(e), y_1 \in F_E^2(e)$ and $F_E^1(e) \cap F_E^2(e) = \phi$ for all $e \in E$. Thus $x_1 \in \tilde{Y} \cap F_E^1 = \tilde{Y} \subset F_E^1, y_1 \in \tilde{Y} \cap F_E^2 = \tilde{Y} \subset F_E^2$ and $\tilde{Y} \cap F_E^1 = \tilde{Y} \cap F_E^2 = \phi$, where, $\tilde{Y} \subset F_E^1, \tilde{Y} \subset F_E^2$ are $(1,2)^*$-soft b-open sets in $\tilde{Y}$. Therefore $(\tilde{Y}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*$-soft $bT_2$-space.

The characterization for $(1,2)^*$-soft $bH$-space is following.

Theorem 3.22. A soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*$-soft $bT_2$-space over $X$ if and only if for distinct points $x_1, y_1$ of $\tilde{X}$, there exists a $(1,2)^*$-soft b-point set $F_A$ containing $x_1$ but not $y_1$ such that $y_1 \notin (1,2)^*$-sbcl($F_A$).

Proof. Let $x_1$ and $y_1$ be two distinct soft points in $(1,2)^*$-soft $bT_2$-space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$. Then there exists disjoint $(1,2)^*$-soft b-open sets $F_A$ and $F_B$ such that $x_1 \in F_A$ and $y_1 \in F_B$. This implies that $x_1 \in G_B \subset F_A$ and $F_B \subset G_B$ is a $(1,2)^*$-soft b-closed set containing $x_1$ but not $y_1$ and $(1,2)^*$-sbcl($F_A$) = $F_A$. Hence $y_1 \notin (1,2)^*$-sbcl($F_A$).

On the other hand, let $x_1$ and $y_1$ be two distinct soft points in $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$. Then there exists a $(1,2)^*$-soft b-point set $F_A$ containing $x_1$ but not $y_1$ such that $y_1 \notin (1,2)^*$-sbcl($F_A$). This implies that $y_1 \notin [((1,2)^* - \text{sbcl}((F_A))) \subset F_A$. Hence $F_A$ and $[(1,2)^* - \text{sbcl}((F_A))] \subset$ are two disjoint $(1,2)^*$-soft b-open sets containing $x_1$ and $y_1$, respectively. Thus $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is a $(1,2)^*$-soft $bT_2$-space over $X$.

Theorem 3.23. Let$(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*$-soft $bT_2$-space over $X$ and $x_1 \in X$. Then every soft singleton $\{x_1\}$ is a $(1,2)^*$-soft $b$-closed.

Proof. Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a $(1,2)^*$-soft $bT_2$-space over $X$. Let $x_1, y_1 \in \tilde{X}$ and $x_1 \neq y_1$, then there exists $(1,2)^*$-soft b-open sets $F_{E_1}$ and $F_{E_2}$ such that $x_1 \in F_{E_1}, y_1 \in F_{E_2}$ and $F_{E_1} \cap F_{E_2} = \phi$. Since $F_{E_1}$ is a $(1,2)^*$-soft b-open set containing $y_1$ such that $F_{E_2}$ does not contain $x_1$ or $F_{E_2}$ does not contain any other soft point of $\{x_1\}$. Hence a soft point $y_1$ of $\tilde{X}$
distinct from \( x_e \) cannot be a \((1,2)^*\)-soft b-limit point of \( \{x_e\} \). Hence \((1,2)^*\)-soft b-derived set of \( x_e \) is \((1,2)^*\)-sbcl \( \{x_e\} = \phi \) and since \((1,2)^*\)-sbcl \( \{x_e\} = \{x_e\} \cup (1,2)^*\)-sbcl \( \{x_e\} = \{x_e\} \cup \phi = \{x_e\} \). Hence \( \{x_e\} \) is \((1,2)^*\)-soft b-closed.

**Proposition 3.24.** Every \((1,2)^*\)-soft \( bT_2 \)-space is \((1,2)^*\)-soft \( bT_1 \)-space.

**Proof.** Let \( (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \) be a \((1,2)^*\)-soft \( bT_2 \)-space. Then, for every \( x_e, y_e \in \tilde{X} \) and \( x_e \neq y_e \), there exist \((1,2)^*\)-soft b-open sets \( F_{E_1} \) and \( F_{E_2} \) of \( x_e \) and \( y_e \) such that \( F_{E_1} \cap F_{E_2} = \phi \), \( x_e \notin F_{E_1} \Rightarrow x_e \notin F_{E_2} \) and \( F_{E_1} \cap F_{E_2} = \phi \), similarly, \( y_e \notin F_{E_2} \). This implies that \( y_e \notin F_{E_1} \). Hence, \( x_e \notin F_{E_1} \) but \( y_e \notin F_{E_2} \). Therefore, the soft bitopological space \( (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \) is a \((1,2)^*\)-soft \( bT_1 \)-space.

The converse of the above proposition is not true is shown in the following Example.

**Example 3.25.** Let \( X = \{x, y, z\}, E = \{e_1\} \) the soft subsets of \( X \) is given as in the Example 3.10. Define \( \tilde{\tau}_1 = \{\tilde{X}, \phi, G_{E_1}\} \) and \( \tilde{\tau}_2 = \{\tilde{X}, \phi, G_{E_2}\} \). Then \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) is a soft bitopological space. The collection of \((1,2)^*\)-soft b-open sets are \((\tilde{X}, \phi, G_{E_1}, G_{E_2})\). Then \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) is a soft bitopological space. The collection of \((1,2)^*\)-soft b- close sets are \((\tilde{X}, \phi, G_{E_1}, G_{E_2})\) and \((1,2)^*\)-soft b-closed sets are \((\tilde{X}, \phi, G_{E_1}, G_{E_2})\) and \((1,2)^*\)-soft b-closed set is \((\tilde{X}, \phi, G_{E_1}, G_{E_2})\). Then this soft bitopological space is \((1,2)^*\)-soft \( bT_1 \)-space. Since every soft singleton set is \((1,2)^*\)-soft b-closed set.

Consider the soft points \((e_1, \{x\}), (e_1, \{z\}) \notin \tilde{X} \) and \((e_1, \{x\}) \neq (e_1, \{y\}) \); there does not exists disjoint \((1,2)^*\)-soft b-open sets. Then \((\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)\) is not a \((1,2)^*\)-soft \( bT_2 \)-space.

**Theorem 3.26.** Let \( \tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\sigma}_1, \tilde{\sigma}_2, E) \) be a bijective \((1,2)^*\)-soft b-open mapping and if \( \tilde{X} \) is a \((1,2)^*\)-soft \( bT_2 \)-space, then \( \tilde{Y} \) is a \((1,2)^*\)-soft \( bT_2 \)-space.

**Theorem 3.27.** Let \( \tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E) \rightarrow (\tilde{Y}, \tilde{\sigma}_1, \tilde{\sigma}_2, E) \) a injective \((1,2)^*\)-soft b- irresolute mapping and if \( \tilde{Y} \) is a \((1,2)^*\)-soft \( bT_2 \)-space, then \( \tilde{X} \) is a \((1,2)^*\)-soft \( bT_2 \)-space.

References

[10] Revathi and K.Bageerathi, *(1,2)^*-soft b-continuous and (1,2)^*-soft b-closed map*, communicated.